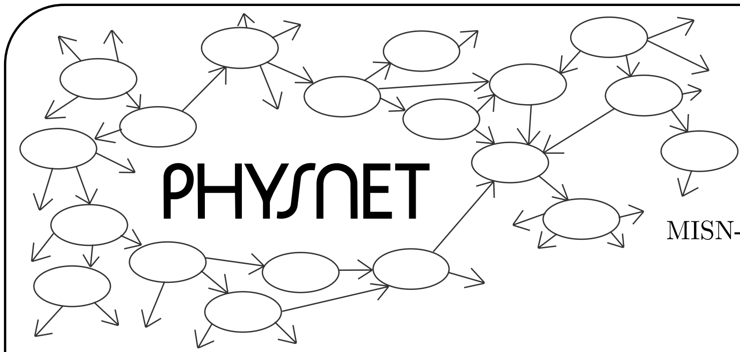
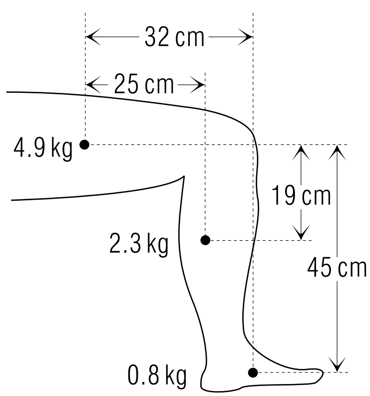


PHYSNET

MISN-0-6

STATIC EQUILIBRIUM AND THE CENTERS OF GRAVITY AND MASS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

STATIC EQUILIBRIUM AND THE CENTERS OF GRAVITY AND MASS

by

Peter Signell and Charles Lavine

Title: **Static Equilibrium and the Centers of Gravity and Mass**

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Version: 6/11/2002 Evaluation: Stage 0

Length: 1 hr; 36 pages

Input Skills:

1. Integrate functions (MISN-0-1).
2. Add and multiply vectors (MISN-0-2).
3. State the equilibrium condition for an object acted on by a set of concurrent forces (MISN-0-5).
4. Define torque (MISN-0-5).

Output Skills (Knowledge):

K1. State the equilibrium conditions for an extended object that is acted upon by any set of forces, concurrent or non-concurrent.

K2. Define the center of gravity of an object.

K3. Define the center of mass of a system of point particles or of an object.

Output Skills (Problem Solving):

S1. Starting with the force diagram for an extended body use the conditions for static equilibrium to determine the magnitude and/or direction of some unknown forces on the body when the other forces are given.

S2. Calculate the location of the center of mass of a given system of point masses.

S3. Extend S2 to include systems of discrete extended masses having known centers of mass.

S4. Given a mass density and a simple geometrical shape for a continuous object, locate the center of mass of the object using integration.

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STATIC EQUILIBRIUM AND THE CENTERS OF GRAVITY AND MASS

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Peter Signell and Charles Lavine

1. Introduction

1a. Conditions for Motionlessness. An important question, with widespread application in physics and engineering, is: Under what conditions will a motionless system not tend to start moving in any direction or rotating about any axis; that is, what are the conditions for "static equilibrium?" The answer is: When the resultants of the forces and the torques acting on the system are zero.

1b. Dealing With Extended Objects. The objects considered in this unit are extended objects; that is, they have finite size in contrast to mathematically idealized point objects. The study of extended objects, acted upon by gravity and able to rotate, brings to mind this question: How do we take into account the weight of each infinitesimable part of the object? We certainly do not want to draw a vector force diagram for each infinitesimal element and then compute the torque on each such element. The answer is to use the rather simple concept of the object's "Center of Gravity," which, for objects small compared to the earth, is just the object's Center of Mass. We will explore these concepts at some length and develop their use in solving static equilibrium problems.

2. Application to Extended Objects

2a. Zero Resultant Force and Torque. For static equilibrium to occur it is not enough that the sum of all of the forces on an object be zero. This can be seen by referring to the object shown in Fig. 1. The object is not in equilibrium in spite of the fact that the net force is zero. Two forces act on the object. They are equal in magnitude and oppositely directed so the resultant force is zero. Because the forces are not along the same line, their effect is to tend to rotate the object (clockwise for the forces shown in Fig. 1).

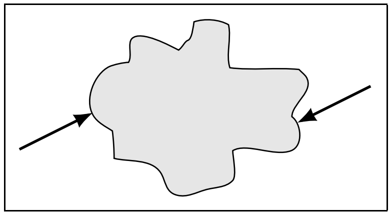


Figure 1. Two equal but opposite forces that tend to rotate an object.

The system in Fig. 1 is not in static equilibrium because the forces produce a net torque which will be accompanied by rotational acceleration. Complete equilibrium will occur only when the net force and the resultant torque, relative to any point whatever, are both zero. Hence, with forces $\vec{F}_1$, $\vec{F}_2$, …, $\vec{F}_n$ producing torques $\vec{\tau}_1$, $\vec{\tau}_2$, …, $\vec{\tau}_n$ relative to some arbitrary fixed point, the conditions for equilibrium are that the resultant force, $\vec{F}_R$, and the resultant torque, $\vec{\tau}_R$, are both zero:

$$\vec{F}_R \equiv \sum_{i=1}^{n} \vec{F}_i \equiv \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_n = 0\,, \tag{1}$$

$$\vec{\tau}_R \equiv \sum_{i=1}^{n} \vec{\tau}_i = \vec{\tau}_1 + \vec{\tau}_2 + \ldots + \vec{\tau}_n = 0\,. \tag{2}$$

2b. Using Vector Components. In finding solutions to problems, component equations are often found to be more useful than vector equations such as Eqs. (1) and (2). To find the Cartesian x-component equations equivalent to Eqs. (1) and (2), we simply multiply each of those equations by the x-coordinate unit vector, $\hat{x}$, and obtain the x-components of the resultant force and torque vectors:

$$F_{Rx} = \vec{F}_R \cdot \hat{x} = \sum_{i=1}^{N} (\vec{F}_i \cdot \hat{x}) = \sum_{i=1}^{N} F_{ix} = 0\,, \tag{3}$$

$$\tau_{Rx} = \vec{\tau}_R \cdot \hat{x} = \sum_{i=1}^{N} (\vec{\tau}_i \cdot \hat{x}) = \sum_{i=1}^{N} \tau_{ix} = 0\,, \tag{4}$$

where F_{ix} indicates the x-component of the i^{th} force. There are similar equations for the y- and z-components, obtained through multiplying by the unit vectors $\hat{y}$ and $\hat{z}$ rather than by $\hat{x}$. Note that a force or torque *component* may be either positive or negative depending upon the direction of the force in relation to the direction of the corresponding coordinate axis.

3. The Center of Gravity

3a. When Gravity is the Force. The weight of an object is a force due to the gravitational attraction between the matter in the object and the matter in the earth. This gravitational pull should not be thought of as a single force between the earth and the body as whole. Each submicroscopic particle that makes up the body experiences a gravitational force due to the presence of the large amount of matter comprising the earth. It is the sum of all these nearly parallel forces that make up the resultant force we call the weight of the object. The point at which this single force, equal to the resultant force, must act to have the same effect as all of the forces between the constituent particles and the earth is called the "center of gravity" (hereafter CG).

3b. The Center of Gravity. When no forces other than gravity are acting on an object, we can always find a point called the object's "Center of Gravity" where, for purposes of static equilibrium, the object's mass may be considered to be concentrated. By this we mean that we can replace the actual object by a weightless rigid structure which has the same shape as the object but whose weight occurs entirely at the object's Center of Gravity. An object's Center of Gravity is a single point whose position can be calculated from the distribution of weight within the actual object. Once the position of the Center of Gravity, $\vec{r}_{CG}$, has been determined, the resultant gravitational torque on the object is given by (see this module's Appendix A):

$$\vec{\tau}_R = \vec{r}_{CG} \times W\hat{g}\,, \qquad (5)$$

where $\hat{g}$ is a unit vector pointing downward, toward the center of the earth, and W is the weight of the object (the weight of the object is the magnitude of the gravitational force on the object). If the object is to be in static equilibrium, the resultant torque must be zero. Solving a static equilibrium problem for an extended object usually begins with determining the location of the object's Center of Gravity. Note that we have not specified any particular origin for computing the vector to the Center of Gravity, so that origin could be anywhere and the equation would still be valid. In general, the origin is chosen as a point about which the system would naturally rotate, such as the point of suspension of the system or a point at which the system is in contact with the ground or with a road.

3c. Locating the Center of Gravity Experimentally. To experimentally determine the center of gravity of an object we suspend it from

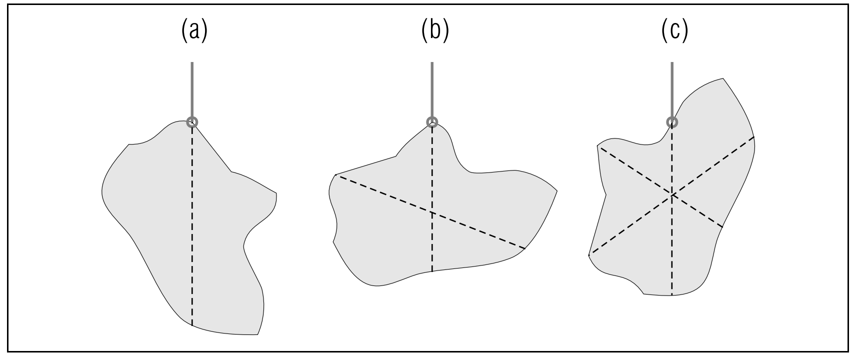


Figure 2. Locating the Center of Gravity of an object by drawing vertical lines from several different points of suspension.

any point on its surface and draw a vertical line through the object from that point downward, as in Fig. 2a. We then pick some different point on the surface of the object and repeat the procedure, as in Fig. 2b. The point where the two lines cross is the Center of Gravity. If we repeat the procedure again, as in Fig. 2c, all three lines will cross at the Center of Gravity. Any number of such lines can be drawn and the lines will all cross each other at the Center of Gravity *(derived in Appendix C; Appendices A and B are prerequisites)*.

▷ Cut out an oblong piece of cardboard or wood and suspend it from an edge using a thread or string as appropriate for its weight. Then: (1) carry out the above-outlined procedure and see for yourself that all of the lines you draw on the object, as you suspend it from diffent points on its edge, really do cross each other at the same point; and (2) suspend the object from your newly found center of gravity by poking a needle or pencil through the point of intersection and show that the object has no preferred rotational orientation about this point.

3d. Locating the Center of Gravity by Calculation. The vector to the Center of Gravity of a system of N particles, for cases where the system is small compared to the size of the earth, can be calculated to high accuracy from this equation for the system's Center of Mass *(derived in Appendix A)*:

$$\vec{r}_{CG} = \vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{N} m_i\,\vec{r}_i \qquad \text{(system small compared to earth)}\,, \qquad (6)$$

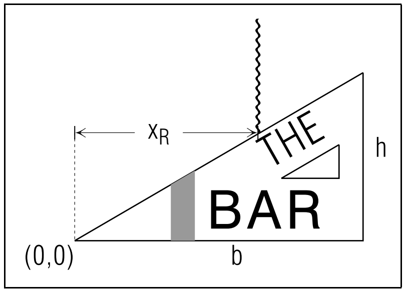


Figure 3. Determining the equilibrium point of suspension of a tringular sign.

where the $i^{\rm th}$ particle has mass m_i and position vector $\vec{r}_i$. For an extended object we have a similar equation *(derived in Appendix B; Appendix A is prerequisite)*:

$$\vec{r}_{CG} = \vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm(\vec{r}) \qquad \text{(object small compared to earth)}\,, \quad (7)$$

where the integral is over the volume of the extended object.

4. A Calculational Example

The owner of the Triangle Bar wishes to erect a sign (of triangular shape, naturally) from a single point, as shown in Fig. 3. The owner needs to find the distance x_R at which the sign will balance. The sign is exceedingly small compared to the size of the earth and it is an extended object so we use Eq. (7).

We start by taking the x-component of Eq. (7):

$$x_{CG} = x_{CM} = \frac{1}{M}\int x\, dM\,. \qquad (8)$$

To convert to an integral over space, we define the surface density of the sign, its mass per unit area, as σ so that $M = \sigma\, A$ and $dM = \sigma\, dA$. Here

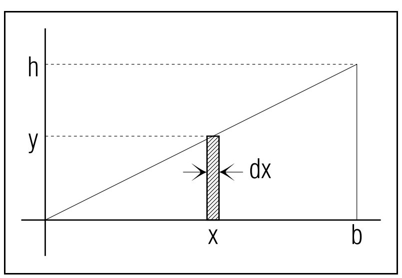


Figure 4. An element of area with constant x for determining x_{CM} for the sign in Fig. 3.

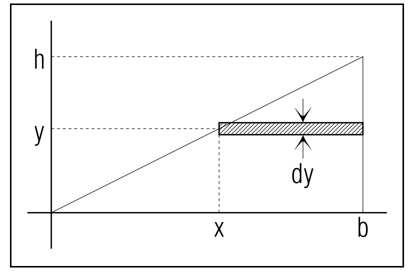


Figure 5. An element of area with constant y for determining y_{CM} for the sign in Fig. 3.

A is the area of the sign. Then we can rewrite Eq. (8) as:

$$x_{CM} = \frac{1}{A}\int x\, dA\,. \qquad (9)$$

We now interpret dA to be all parts of the area that have the same value for x. An example is shown as the shaded area in Fig. 4. The size of the area is $dA = y\, dx$ and all parts of it have the same value for x. Therefore the integral over all area becomes:

$$x_{CM} = \frac{1}{A}\int_0^b x\, y(x)\, dx\,. \qquad (10)$$

We interpret this integral as: "We weight each element of area, $dA = y(x)\, dx$, with its value of x, and integrate over the whole area." Now $y(x) = hx/b$ and $A = bh/2$ so we get:

$$x_{CM} = \frac{2}{bh}\int_0^b x\, \frac{hx}{b}\, dx = \frac{2}{b^2}\int_0^b x^2\, dx = \frac{2}{3}\, b\,, \qquad (11)$$

hence:

$$x_{CG} = \frac{2}{3}\, b\,.$$

▷ Show that the y-position of the Center of Gravity is at $h/3$. *Help: [S-2]*

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.Jules Kovacs made useful suggestions.

A. Centers of Gravity and Mass for a Particle System

We treat the case of a system of point particles near the surface of the earth and we assume that gravity is the only force acting on the particles in the system. This means that the force on each particle in the system is just the force of gravity, the particle's weight. We write the weight of the $i^{\rm th}$ particle as $\vec{w}_i$, whereupon the gravitational torque on the $i^{\rm th}$ particle becomes:

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_i = \vec{r}_i \times \vec{w}_i \,. \qquad (12)$$

Then Eq. (4) for the resultant torque on the system is:

$$\vec{\tau}_R = \sum_{i=1}^{N} \vec{\tau}_i = \sum_{i=1}^{N} \vec{r}_i \times \vec{w}_i \,. \qquad (13)$$

The weights are all forces in the downward direction, toward the center of the earth. We denote that direction with the unit vector $\hat{g}$ so $\vec{w}_i = w_i\hat{g}$ and we have:

$$\vec{\tau}_R = \sum_{i=1}^{N} \vec{r}_i \times \vec{w}_i = \sum_{i=1}^{N} \vec{r}_i \times w_i\hat{g} = \left(\sum_{i=1}^{N} \vec{r}_i w_i\right) \times \hat{g}\,, \qquad (14)$$

where we have factored out the cross product with $\hat{g}$ because it is common to all terms in the sum. The vector sum in the parenthesis in Eq. (14) is just the sum of the position vectors to the individual particles but with each position vector weighted by the weight of the particle. This vector sum, divided by the weight of the system, is the "Center of Gravity" of the system:

$$\vec{r}_{CG} = \frac{1}{W} \sum_{i=1}^{N} w_i\, \vec{r}_i \,. \qquad (15)$$

Finally, we rewrite Eq. (15) to emphasize that each particle's position vector is weighted by the particle's fraction of the system's weight:

$$\vec{r}_{CG} = \sum_{i=1}^{N} \frac{w_i}{W}\, \vec{r}_i \,. \qquad (16)$$

Using Eq. (16), Eq. (14) can be written:

$$\vec{\tau}_R = \vec{r}_{CG} \times W\hat{g}\,. \qquad (17)$$

This demonstrates that the resultant gravitational torque on a system of particles is exactly the same as what would be produced by a single point particle weighing W, located at the system's Center of Gravity.

We can write the weight of each contributing particle as its mass times the acceleration of gravity, g, and the total weight of the system, W, as g times the total mass of the system, M. Then Eq. (16) becomes:

$$\vec{r}_{CG} = \frac{1}{M} \sum_{i=1}^{N} m_i\, \vec{r}_i \qquad \text{(system small compared to earth)}\,. \qquad (18)$$

For an ordinary-size system near the surface of the earth, this equation for the Center of Gravity, Eq. (19), no longer contains any reference to gravity. In fact, it is the equation that defines the position vector to the "Center of Mass" of the system, defined by giving each particle's position vector a weight equal to its fraction of the mass of the system of particles:[1]

$$\vec{r}_{CM} \equiv \frac{1}{M} \sum_{i=1}^{N} m_i\, \vec{r}_i \,. \qquad (19)$$

Then one can write:

$$\vec{r}_{CG} = \vec{r}_{CM} \qquad \text{(system small compared to earth)}\,, \qquad (20)$$

B. Centers of Gravity and Mass for an Object

Any extended object can be treated as if it is composed of many infinitesimal objects, each having an infinitesimal mass dm that contributes to Eq. (16). Then the sum in Eq. (16) becomes an integral over the entire object:[2]

$$\vec{r}_{CG} = \frac{1}{M} \int \vec{r}\, dm(\vec{r}) \qquad \text{(object small compared to earth)}\,, \qquad (21)$$

Here the integral is over the mass of each infinitesimal element of the extended object. The infinitesimal mass of the infinitesimal element at position $\vec{r}$ has been written as $dm(\vec{r})$. Equation (21) is actually the definition of an extended object's Center of Mass:

$$\vec{r}_{CM} \equiv \frac{1}{M} \int \vec{r}\, dm(\vec{r})\,,$$

[1] Note that in this sentence the word "weight" is used in the mathematical or statistical sense, not in the gravitational sense.

[2] The conversion of a sum over discrete objects to an integral over infinitesimal parts of an extended object is developed in "Simple Differentiation and Integration" (MISN-0-1).

so one can write:

$$\vec{r}_{CG} = \vec{r}_{CM} \qquad \text{(object small compared to earth)}\,, \qquad (22)$$

Note that Eq. (17) is the same as Eq. (22) so the same equation is valid for both extended systems and discrete ones.

One can multiply Eq. (21) by the unit vector in the x-direction and get the x-component of the vector to the Center of Gravity:

$$x_{CG} = \frac{1}{M}\int x\,dm\,. \qquad \text{(object small compared to earth)}\,, \qquad (23)$$

and there are similar equations for the y- and z-components, obtained by using $\hat{y}$ and $\hat{z}$ in place of $\hat{x}$.

C. The Center of Gravity is Below a Suspension Point

The reason why a vertical line drawn downward from any point of suspension point passes through the suspended object's the Center of Gravity can be easily seen in Fig. 6. The figure shows an object suspended from a point on its edge, but rotated so it is unbalanced. By this we mean that if we let it go it will start swinging downward, rotating about the point of suspension. The reason it starts rotating is because there is a torque on the object equal to the vector product of the Center of Gravity position vector and the weight vector of the object.[3] In terms of magnitude:

$$\tau_R = |\vec{r}_{CG} \times W\,\hat{g}| = r_{CG}\,W\,\sin\theta\,.$$

where θ is the angle between the direction of the Center of Gravity position vector and the downward direction of the force of gravity (see Fig. 6), W

[3]See Sect. 3 for the reason this is correct.

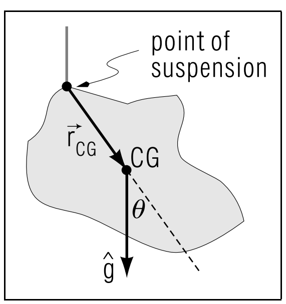


Figure 6. The object will experience a torque about the suspension point, due to gravity, unless $\theta = 0$.

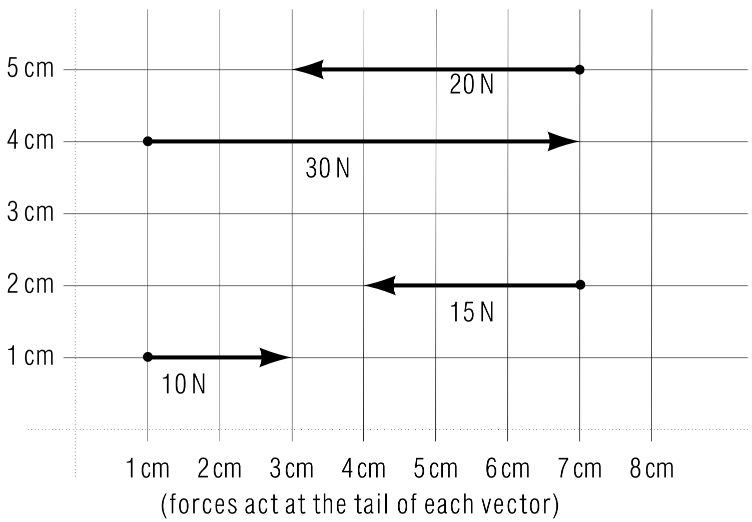
is the weight of the object, $\vec{r}_{CG}$ is the distance from the poi. There is no torque on the object only if $\theta = 0$ and that is the case if we just let the object hang downward without rotation. In such a case, a vertical line drawn downward from the point of suspension will certainly pass through the Center of Gravity, no matter what suspension point is used.

PROBLEM SUPPLEMENT

Note 1: There is a lot of general problem-solving help on the current topic in this module's *Special Assistance Supplement*. We suggest you look over that material before trying to solve the problems below.

Note 2: Problems 2, 10, and 11 also occur in this module's *Model Exam*.

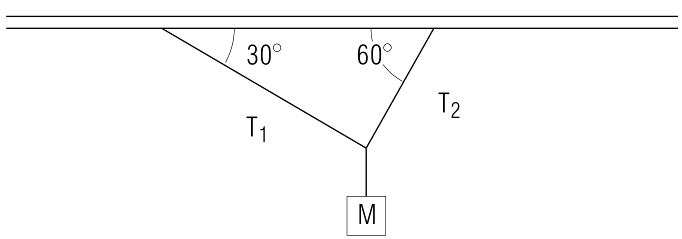
1.



a. Find the resultant force of this set of parallel forces.

b. Find the resultant torques produced by this set of forces around the points $(x,y) = (0\,\text{cm},0\,\text{cm})$, $(x,y) = (1\,\text{cm},3\,\text{cm})$, and $(x,y) = (1\,\text{cm},4\,\text{cm})$.

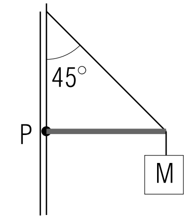
2.



Determine the tensions T_1 and T_2 in the ropes if the mass M weighs

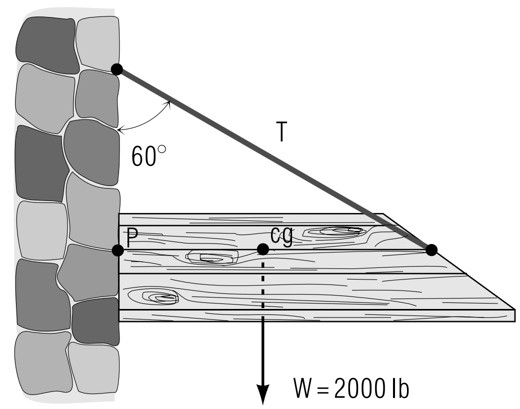
10 lb.

3.



Determine the force exerted on the beam by the wall at point P if the mass M weighs 20 lb and the weights of the cable and beam are negligible.

4.

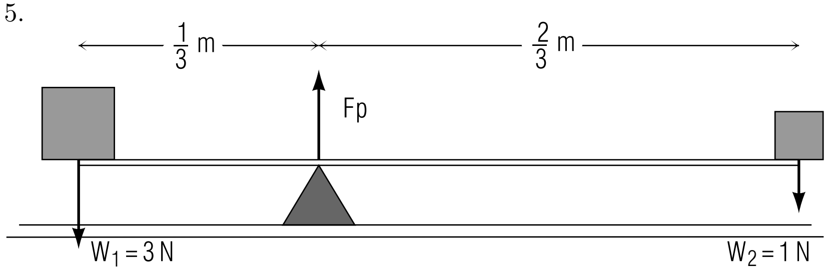


A drawbridge weighing 2000 lb is suspended from the side of a castle by a cable, as shown above.

a. Find the tension in the cable.

b. Find the force exerted on the bridge by the wall of the building at point P. (Hint: you can consider the weight of the drawbridge to be acting on its center of gravity at a point halfway along its length.) *Help: [S-1]*

5.

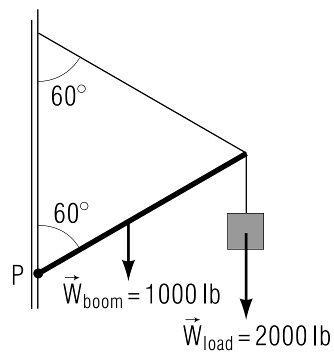


A uniform meterstick with two different weights on it is balanced on a pivot as shown in the sketch. Assume that the weight of the meterstick acts upon its center of gravity, which is located at the middle of the meter stick.

a. Find the weight of the meterstick, $W_{\rm ms}$, in newtons.

b. Find the force F_p exerted by the pivot upon the meterstick.

6.



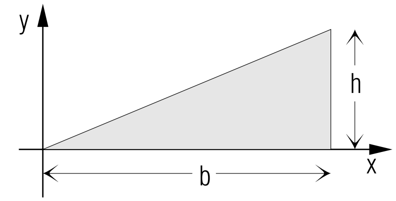
A load of weight 2000 lb is suspended by a cable and boom from the side of a building, as shown in the sketch. The boom weighs 1000 lb. The center of gravity of the boom is halfway along its length.

a. Draw a one-body force diagram showing all the forces acting on the boom.

b. Find the tension in the cable.

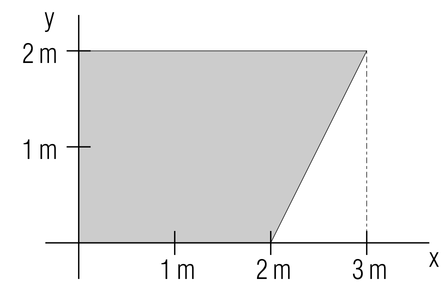
c. Find the force exerted by the wall on the boom at the pivot point P.

7.



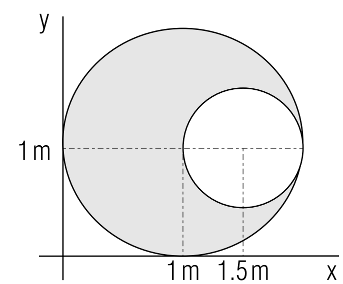
Find the center of mass of this right triangle with base b and height h, assuming it has a constant surface mass density σ. (Hint: $dM = \sigma\, dy\, dx$).

8.



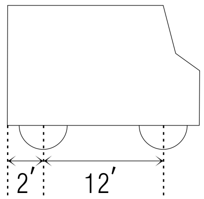
Find the center of mass of the figure, assuming it has a constant surface mass density.

9.



Find the center of mass of this figure assuming it has a constant surface mass density.

10.



A van is loaded so that the load on each pair of wheels, front and back, is the same: 2400 pounds. The rear axle is 12 feet behind the front axle. The rear overhang of the van extends two feet behind the rear axle. If an additional weight of 600 pounds is loaded onto the very back of the van what is the new load distribution of the front and rear tires?

(Be sure to draw a correct one-body force diagram showing all of the forces acting on the object under consideration.)

11. The mass of the earth is 5.98×10^{24} kilograms while the mass of the moon is 7.34×10^{22} kilograms. The average earth-moon separation (center-to-center) is 3.84×10^{8} meters. Find the location of the Center of Mass of the earth-moon system.

Brief Answers:

1. a. $F_R = 5\,\mathrm{N}$, to the right

   b. (0 cm,0 cm): $\tau_R = 0\,\mathrm{N\,m}$
   (1 cm,3 cm): $\tau_R = 15\,\mathrm{N\,cm}$, out of the page
   (1 cm,4 cm): $\tau_R = 20\,\mathrm{N\,cm}$, out of the page
   (Torques all different because resultant force is not zero).

2. $T_1 = 5\,\mathrm{lb}$; $T_2 = 8.7\,\mathrm{cm}$.

3. $F_{\mathrm{wall}} = 20\,\mathrm{lb}$ to the right.

4. a. $T = 2000\,\mathrm{lb}$. (NOT $2000\,\mathrm{lb}/\cos 60°$) *Help: [S-1]* [4]

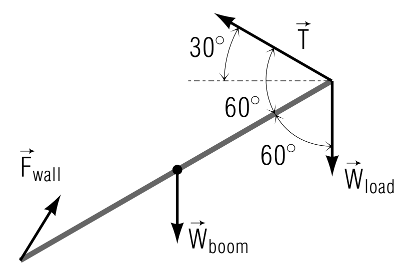
   b. $F_{\mathrm{wall},x} = 1000\sqrt{3}\,\mathrm{lb.} = 1.7 \times 10^3\,\mathrm{lb}$
   $F_{\mathrm{wall},y} = 1000\,\mathrm{lb}$.

   Magnitude is 2000 lb, directed at an angle of 30° above horizontal to the right.

5. a. $W_{\mathrm{m.s.}} = 2\,\mathrm{N}$.

   b. $F_p = 6\,\mathrm{N}$.

6. a.



   b. $T = 2500\,\mathrm{lb}$.

   c. $F_{\mathrm{wall},x} = 1250\sqrt{3}\,\mathrm{lb}$; $F_{\mathrm{wall},y} = 1750\,\mathrm{lb}$; Magnitude is 2783.88 lb, directed at angle 38.95° above horizontal to right.

7. $x_{\mathrm{CM}} = 2/3\,b$; $y_{\mathrm{CM}} = 1/3\,h$.

8. $x_{\mathrm{CM}} = 19/15\,\mathrm{m}$; $y_{\mathrm{CM}} = 16/15\,\mathrm{m}$.

9. $x_{\mathrm{CM}} = 5/6\,\mathrm{m}$; $y_{\mathrm{CM}} = 1.0\,\mathrm{m}$.

[4] See sequence [S-1] near the end of this module's *Special Assistance Supplement*.

10. $F_f = 2300\,\text{lb}$

    $F_r = 3100\,\text{lb}.$

11. 4.66×10^6 m from earth's center.

SPECIAL ASSISTANCE SUPPLEMENT

1. The Equilibrium Conditions

1a. Statement. Can you understand the two conditions that must exist for an extended body to be in static equilibrium in a given force environment?

1b. Properties. Remember that force and torque are both vector quantities. In what direction is the torque associated with a given force and reference moment arm? Because of the vector nature of force and torque, the two equilibrium conditions in vector notation actually become six scalar equations.

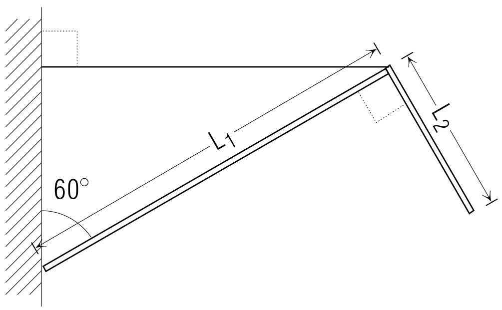
If forces are confined to a plane (say the x-y plane) then only three equations are useful, so only three unknowns may be found. Can you demonstrate this to yourself?

2. Application of the Conditions

2a. Strategies for Problem Solving.

1. Read the problem carefully so that you understand what is given and what is asked for.
2. Draw a force diagram for the object in question. Include a representation of all forces both known and unknown. Remember that, in general, it takes three numbers to express both the magnitude and direction of a force. You may denote the three force components (F_x,F_y,F_z), or as the magnitude of the force and two angles that specify its direction, etc. Sometimes one force component is obviously zero and need not be considered further. Don't forget forces, such as gravity, which are imposed without contact. Note that the force of gravity upon an object can always be considered as acting upon the center of gravity (or the center of mass) of the object.
3. If you wish a complete solution (values found for all unknowns) you need as many independent equations as unknowns. Using the equilibrium conditions as expressed in component form, write the necessary equations. It is usually possible to obtain simple torque equations by carefully choosing the point about which the torques are computed.
4. Solve the set of equations for the desired unknown forces.
5. Check to see that your results are reasonable, units correct, etc.

2b. An Example.



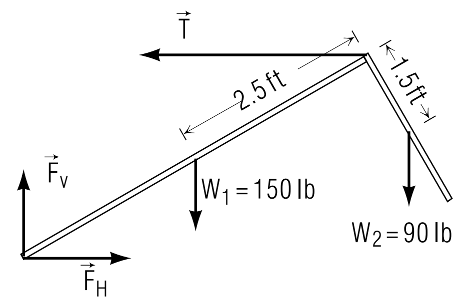
Let us apply these problem solving strategies to an example. An L-shaped

object consists of two very thin uniform beams joined rigidly so that they make an angle of 90° with each other. One end of one of the beams is pinned to a vertical wall making a 60° angle with the wall.

The point where the beams are joined is also tied to the vertical wall by means of a horizontal cable (see sketch). The beam of length $L_1 = 5$ ft, weighs 150 pounds, while the beam of length $L_2 = 3$ ft, weighs 90 pounds. The system is in equilibrium.

The questions we wish to answer are: (a) What is the tension in the cable; and (b) What force does the vertical wall exert on the lower end of beam L_1?

Because the system is in static equilibrium, we know that all of the forces and torques on the object must add to zero, so we use this fact to find the unknown forces. (The center-of-mass of each of the thin uniform beams is in the middle of the beam).



As has been emphasized, the first thing we must do is draw a one-body diagram of the object under consideration (in this case the L-shaped object) and replace each object (the wall, the cable, gravity) in contact with it by the force this object exerts on the L-shaped object.

These are all the forces due to external agents acting upon the L-shaped object. Note that because we didn't know the magnitude or the direction of the force exerted by the wall, we have two unknowns and we've represented these by an unknown horizontal force, $\vec{F}_H$, and an unknown vertical one, $\vec{F}_V$. The conditions for equilibrium are:

(i) all the forces must add to zero (and, since no horizontal force can cancel a vertical one, the horizontal and vertical forces must separately add to zero); and

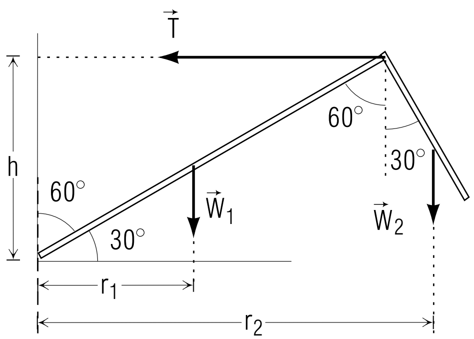
(ii) all the torques about any point (we are free to choose the point) must add to zero. We are free to choose any number of such points because the object has no angular acceleration about *any* point.

2c. Forces Add to Zero. For the forces to add to zero, relative to some fixed Cartesian Coordinate system, all of the components must separately add to zero. If our coordinate directions are (1) the horizontal, and (2) the vertical directions in the plane of the page, and (3) the direction perpendicular to the page; then in the direction perpendicular to the page there are no forces, so we need to consider only the two components in the plane of the page. So,

$$F_H = T \qquad \text{and} \qquad F_V = W_1 + W_2 = 240\,\text{lb}\,.$$

Now we need to find either F_H or T and all forces will be known.

2d. Torques Add to Zero.



Taking torques about the point where F_V and F_H act on our object, the torque due to these two forces are each zero. The torque of T is out of the page, tending to produce clockwise rotation. So for equilibrium the torque of T must equal the sum of the torques of W_1 and W_2.

From the condition that the clockwise and counterclockwise torques balance:

$$hT = r_1 W_1 + r_2 W_2$$

$$r_1 = \frac{L_1}{2}\cos 30^\circ = \sqrt{3}\frac{L_1}{4} = \frac{5\sqrt{3}}{4}\,\text{ft}\,.$$

$$r_2 = L_1 \cos 30^\circ + \frac{L_2}{2}\sin 30^\circ = \left(\frac{5\sqrt{3}}{2} + \frac{3}{4}\right)\,\text{ft}\,.$$

$$h = L_1 \cos 60^\circ = \frac{L_1}{2} = 2.5\,\text{ft}\,.$$

This allows us to solve for T,

$$2.5\,\text{ft}\,T = \frac{5\sqrt{3}}{4}\,\text{ft} \times 150\,\text{lb} + \left(\frac{5\sqrt{3}}{2} + \frac{3}{4}\right)\,\text{ft} \times 90\,\text{lb}.$$

$$T = 312.8\,\text{lb} = F_H$$

So all the forces on the L-shaped object are determined. $\vec{F}_V$ and $\vec{F}_H$ may be added vectorially to find the magnitude and direction of the single force the wall exerts on the object.

3. Center of Gravity, Center of Mass

3a. Volume Mass Distribution. The center of gravity and center of mass of an extended object are defined in different ways, but are really the same point.

The position of the center of mass of an object is defined by Text Eq. (**??**):

$$x_{\rm CM} = \frac{1}{M}\int x\,dM$$

$$y_{\rm CM} = \frac{1}{M}\int y\,dM$$

$$z_{\rm CM} = \frac{1}{M}\int z\,dM,$$

or, in a more compact form,

$$\vec{r}_{\rm CM} = \frac{1}{M}\int \vec{r}\,dM.$$

The integrals are taken over the entire mass of the object. We can make the above equations easier to grasp if we replace the "infinitesimal mass element" dM by the more physical infinitesimal volume element $dV = dx\,dy\,dz$. If the mass density of the object is $\rho(\vec{r}) = \rho(x, y, z)$,[5] in units of mass per unit volume (e.g., kg/m^3) then the (infinitesimal) amount of mass contained in the (infinitesimal) volume element dV around the point $\vec{r}$ is

$$dM = \rho\,(\vec{r})\,dV,$$

[5] ρ is a function of position—we allow the mass density to vary across the object.

and so our definition of the center of mass becomes

$$x_{\rm CM} = \frac{1}{M}\int\int\int x\,\rho(x,y,z)\,dx\,dy\,dz$$

$$y_{\rm CM} = \frac{1}{M}\int\int\int y\,\rho(x,y,z)\,dx\,dy\,dz$$

$$z_{\rm CM} = \frac{1}{M}\int\int\int z\,\rho(x,y,z)\,dx\,dy\,dz,$$

or, in a more compact form,

$$\vec{r}_{\rm CM} = \frac{1}{M}\int \vec{r}\,\rho(\vec{r})\,dV.$$

The integrals are now over the volume of the object, i.e., the shape of the object determines the limits on the integrals.

Since M is the mass of the whole object, $\rho(\vec{r})$ is related to M by

$$M = \int \rho(\vec{r})\,dV.$$

That is, M is the integral of ρ over the whole volume of the object.

3b. Surface Mass Distribution. A good many center-of-mass problems involve two-dimensional objects, which make the integrations a good deal simpler. If the surface mass density $\sigma(x,y)$, in units of mass per unit area (e.g. kg/m^2) is given, then the position of the center of mass is given by:

$$x_{\rm CM} = \frac{1}{M}\int\int x\,\sigma(x,y)\,dx\,dy$$

$$y_{\rm CM} = \frac{1}{M}\int\int y\,\sigma(x,y)\,dx\,dy,$$

where the integrals are over the entire area of the object, and

$$M = \int\int \sigma(x,y)\,dx\,dy$$

relates the total mass M to the surface mass density $\sigma(x,y)$.

3c. Constant Surface Density. Most such two-dimensional problems involve a further simplification - an assumption that the surface mass density is a constant, rather than a varying function of position. The only tricky part of the problem lies in picking the right limits for the integrals. Problem 7 in the Problem Supplement is of just this type. The surface mass density σ is a constant, so the total mass of the object is

$$M_{\rm object} = \sigma\int\int dx\,dy = \frac{1}{2}bh\sigma,$$

since $b\,h/2$ is the area of the object. If we choose to integrate over y first, then the limits on the integrals become $y = 0$ to $y = (h/b)x$ for y, and $x = 0$ to $x = b$ for x, because the upper edge of the triangle is the line $y = (h/b)x$, the right edge is the line $x = b$, and the bottom edge is the line $y = 0$. Therefore, the position of the center of mass is given by:

$$x_{\rm CM} = \frac{1}{bh\sigma/2}\int_0^b dx\int_0^{(h/b)x} dy\,x\,\sigma$$

and

$$y_{\rm CM} = \frac{1}{bh\sigma/2}\int_0^b dx\int_0^{(h/b)x} dy\,y\,\sigma$$

The surface mass density σ is a constant and comes outside both integrals, where it cancels out of the problem entirely.

We work out the integrals:

$$\begin{aligned} x_{\rm CM} &= \frac{1}{bh/2}\int_0^b dx\int_0^{(h/b)x} dy\ x \\ &= \frac{2}{bh}\int_0^b dx\,x\,|y|_0^{(h/b)x} \\ &= \frac{2}{bh}\int_0^b dx\,(h/b)x^2 \\ &= \frac{2}{b^2}\left|\frac{1}{3}x^3\right|_0^b \\ &= \frac{2}{3}b, \end{aligned}$$

and

$$
\begin{aligned}
y_{\rm CM} &= \frac{1}{bh\sigma/2}\int_0^b dx \int_0^{(h/b)x} dy\, y \\
&= \frac{2}{bh}\int_0^b dx \left|\frac{1}{2}y^2\right|_0^{(h/b)x} \\
&= \frac{1}{bh}\int_0^b dx\,(h/b)^2 x^2 \\
&= \frac{h}{b^3}\left|\frac{1}{3}x^3\right|_0^b \\
&= \frac{1}{3}h\,,
\end{aligned}
$$

The only conceptual problem was picking the right limits—the rest is ordinary calculus and algebra.

3d. Guessing the CM. We can often guess the position of the center of mass of a simple system. For instance, the center of mass of a sphere with a constant mass density is just the center of the sphere. For any object having a constant mass density, the center of mass is located at the geometric center of the object.

3e. Objects with Holes. Another common type of center-of-mass problem involves objects with "holes" or "chunks" cut out of them. Problems 8 and 9 are of this type. In problem 9, the object is a whole circle with a smaller circular hole cut out of it. We can use a simple trick to solve this type of problem.

Consider the fact that we can "reconstruct" a solid disc by adding together the object we have in Problem 9 with the circular piece that was cut out of the disc in order to make the object. More specifically, the integral of the quantity $\vec{r}\sigma\,dx\,dy$ over the solid disc equals the integral of that quantity over the object plus the integral of the quantity over the circular cutout:

$$\int_{\rm disc}\vec{r}\,\sigma\,dx\,dy = \int_{\rm object}\vec{r}\,\sigma\,dx\,dy + \int_{\rm cut\text{-}out}\vec{r}\,\sigma\,dx\,dy.$$

Furthermore, the centers of mass of each of the three things involved here are:

$$\vec{r}_{\rm CM,disc} = \frac{1}{M_{disc}}\int_{\rm disc}\vec{r}\sigma\,dx\,dy$$

$$\vec{r}_{\rm CM,object} = \frac{1}{M_{\rm object}}\int_{\rm object}\vec{r}\sigma\,dx\,dy$$

$$\vec{r}_{\rm CM,cut\text{-}out} = \frac{1}{M_{\rm cut\text{-}out}}\int_{\rm cut\text{-}out}\vec{r}\sigma\,dx\,dy$$

Combining all of the above equations yields the final result:

$$M_{\rm disc}\,\vec{r}_{\rm CM,disc} = M_{\rm object}\,\vec{r}_{\rm CM,object} + M_{\rm cut\text{-}out}\,\vec{r}_{\rm CM,cut\text{-}out}.$$

The masses are related by

$$M_{\rm disc} = M_{\rm object} + M_{\rm cut\text{-}out}.$$

In the case of problem 9, the positions of the centers of mass of the solid disc and the cutout are obvious:

$$x_{\rm CM,disc} = 1\,{\rm m};\ y_{\rm CM,disc} = 0$$

$$x_{\rm CM,cut\text{-}out} = (1 + 1/2)\,{\rm m} = 3/2\,{\rm m}; y_{\rm CM,cut\text{-}out} = 0\,,$$

which are the positions of the centers of the two circles in question.

Since the object has a constant surface mass density, the masses of the things involved are

$$M_{\rm disc} = \pi\,(1\,m)^2\,\sigma = \sigma\,\pi\,m^2\,,$$

$$M_{\rm cut\text{-}out} = \pi\left(\frac{1}{2}m\right)^2\sigma = \frac{1}{4}\sigma\pi m^2\,,$$

$$M_{\rm object} = M_{\rm whole\ circle} - M_{\rm cut\text{-}out} = \frac{3}{4}\sigma\pi m^2\,.$$

Therefore, using the above handy equation:

$$(\sigma\pi m^2)(1\,{\rm m}) = \left(\frac{3}{4}\sigma\pi m^2\right)x_{\rm CM,object} + \left(\frac{1}{4}\sigma\pi m^2\right)\left(\frac{3}{2}m\right)$$

$$(\sigma\pi m^2)(0) = \left(\frac{3}{4}\sigma\pi m^2\right)y_{\rm CM,object} + \left(\frac{1}{4}\sigma\pi m^2\right)(0)\,,$$

which gives

$$x_{\rm CM,object} = \frac{\sigma\pi m^3 - \frac{3}{8}\sigma\pi m^3}{\frac{3}{4}\sigma\pi m^2} = \frac{5}{6}m\,,$$

and

$$y_{\rm CM,object} = 0\,.$$

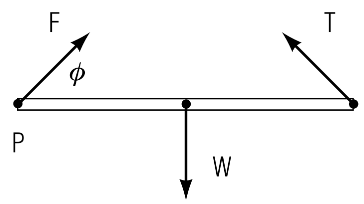
One can use the same technique to solve any similar problem. The equation used in this one particular case can be generalized to cover any object consisting of a total object (abbreviated "total") and a "hole" cutout:

$$M_{\text{total}}\vec{r}_{\text{CM,total}} = M_{\text{object}}\vec{r}_{\text{CM,object}} + M_{\text{cut-out}}\vec{r}_{\text{CM,cut-out}}$$

where:

$$M_{\text{total}} = M_{\text{object}} + M_{\text{cut-out}}\,.$$

S-1 *(from PS-problem 4)*

1. 

   Sketch a one-body diagram of the forces on the drawbridge, giving the force at P an unknown magnitude F and an angle ϕ above the horizontal.
2. Sum horizontal and vertical forces separately.
3. Sum the torques about P so as to eliminate the unknowns, F and ϕ, from the equation. Assign the drawbridge an unknown length ℓ: it drops out.

S-2 *(from TX-Sect. 4c)*

The integral for y_{CM} proceeds similarly to that for x_{CM}, starting with

$$y_{CM} = \frac{1}{M}\int y\,dM\,,$$

and with $dA = x\,dy$ as a strip of area all at the same value for y as in Fig. 5. Then the integral is:

$$y_{CM} = \frac{1}{A}\int_0^h y\,x(y)\,dy\,.$$

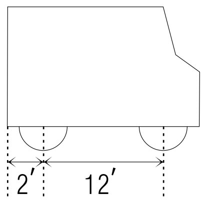
We interpret this integral as: "We weight each element of area, $x(y)\,dy$, with its value of y, and integrate over the whole area."
$x(y) = b - (by/h)$ so:

$$y_{CM} = \frac{2}{bh}\int_0^h y\left(b - \frac{by}{h}\right)dy = \frac{2}{h^2}\int_0^h (hy - y^2)\,dx = \frac{1}{3}\,h\,.$$

MODEL EXAM

1. See Output Skills K1-K3 in this module's *ID Sheet*.

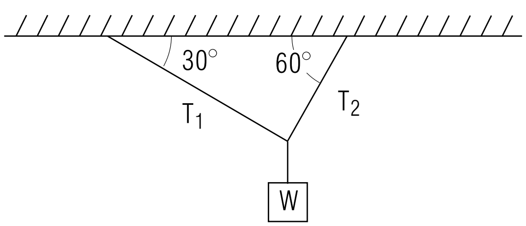
2.



A van is loaded so that the load on each pair of wheels, front and back, is the same: 2400 pounds. The rear axle is 12 feet behind the front axle. The rear overhang of the van extends two feet behind the rear axle. If an additional weight of 600 pounds is loaded onto the very back of the van what is the new load distribution of the front and rear tires?

(Be sure to draw a correct one-body diagram showing all of the forces acting upon the object under consideration.)

3.



Determine the tensions T_1 and T_2 in the ropes if the mass M weighs 10 lb.

4. The mass of the earth is 5.98×10^{24} kilograms while the mass of the moon is 7.34×10^{22} kilograms. The average earth-moon separation (center-to-center) is 3.84×10^{8} meters. Find the location of the center-of-mass of the earth-moon system.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 10.

3. See this module's *Problem Supplement*, problem 2.

4. See this module's *Problem Supplement*, problem 11.