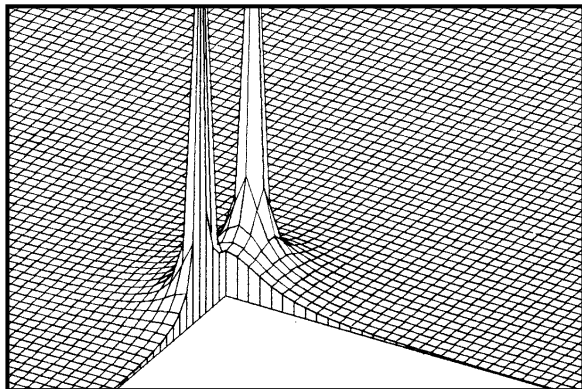


SOME SIMPLE FUNCTIONS IN THE COMPLEX PLANE



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by
Peter Signell

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Title: **Some Simple Functions in the Complex Plane**

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Evaluation: Stage 1

Length: 1 hr; 16 pages

Input Skills:

1. Use the quadratic-root equation to find the zeros of any given quadratic equation (MISN-0-401).
2. Convert from two dimensional Cartesian coordinates to polar, and vice versa (MISN-0-401).
3. Interpret the words “e is the base of natural logarithms” (MISN-0-401).

Output Skills (Problem Solving):

- S1. Determine the complex conjugate, magnitude, and phase of any given complex number.
- S2. Locate the poles of any given reciprocal of a quadratic function.
- S3. Sketch the pole trajectories of any given quadratic reciprocal. Describe the motions of the poles along their trajectories.

Post-Options:

1. “Taylor’s Series for the Expansion of a Function About a Point” (MISN-0-4).
2. “Damped Mechanical Oscillations” (MISN-0-29).
3. “Resonances and Poles; Relationship Between the Real and Imaginary Worlds” (MISN-0-49).
4. “Wave Functions, Probability, and Mean Values” (MISN-0-243).

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1. Introduction

Complex numbers provide one with a very powerful tool in physics. For some subjects, it provides a sort of rich, full view of an object where the real numbers provide only a sort of silhouette. In other cases it provides quick and easy means of analysis, synthesis and solution. Finally, there are cases like the time-dependent Schrödinger where it is an integral part of the theoretical framework.

2. Complex Numbers

2a. Rules for Complex-Number Arithmetic. A complex number z can be defined as an ordered pair of real numbers (x,y) ; e.g., $(2,5)$, $(2\pi, 21.49)$, $(2a + b, c * d)$. The first number in the parenthesis is called the “real part” of the “complex number” z , while the second number in the parenthesis is called the “imaginary” part. What distinguishes complex numbers is the way they combine to form new complex numbers.

Suppose we have two complex numbers, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$. Here are the addition, subtraction, and multiplication rules for those two complex numbers:

$$z_1 \pm z_2 \equiv (x_1 \pm x_2, y_1 \pm y_2) \quad (1)$$

$$z_1 \cdot z_2 \equiv (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \quad (2)$$

2b. “Real” and “Imaginary” Numbers. Complex numbers with no second number $(x,0)$, are referred to as “purely real,” while those with no first member $(0,y)$ are said to be “purely imaginary.” Any complex number can be written as the sum of a real number and an imaginary number, according to the addition rule Eq. (1); e.g.,

$$z = (x, 0) + (0, y) = (x, y).$$

The normal notation and wording for the real and imaginary parts of a complex number are:

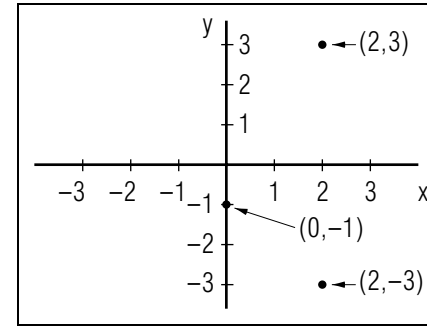


Figure 1. Three complex numbers represented as points on a “complex” plane.

$$\begin{aligned} x &\equiv \text{“real part of } z\text{”} \equiv \text{Re}(z) \\ y &\equiv \text{“imaginary part of } z\text{”} \equiv \text{Im}(z). \\ z &= (\text{Re}(z), 0) + (0, \text{Im}(z)) = (\text{Re}(z), \text{Im}(z)). \end{aligned}$$

2c. The Argand Diagram: A Complex Number Plot. Complex numbers are often represented as points in a two-dimensional complex plane, as illustrated in the Argand¹ diagram of Fig. 1. Here the x -axis is called the “real axis,” the y -axis the “imaginary axis.”

2d. Use of i . In the physical sciences and engineering the combinatorial rules in Eqs. (1)-(2) are remembered and denoted by use of the symbol i , which is treated like any other algebraic symbol except that:

$$i^2 = -1. \quad (3)$$

It is in this sense that we say:

$$i \equiv \sqrt{-1}. \quad (4)$$

Complex numbers are then written:

$$z = x + iy = \text{Re}(z) + i \text{Im}(z) \quad (5)$$

The combinatorial rules of Eq. (1) can be derived from Eq. (4) in a few lines. *Help: [S-1]*

2e. The Complex Conjugate. Only real numbers can correspond to the results of physical measurements, so one must have mechanisms for obtaining real numbers from complex ones. One of the most common ways is to combine a complex number with its “complex conjugate,” either additively or multiplicatively. The complex conjugate of a complex

¹Pronounced “Ar’gänd” in English.

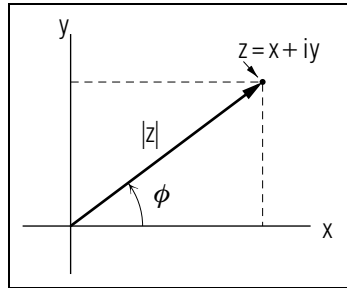


Figure 2. The magnitude $|z|$ of a complex number.

number z is written z^* , and is defined as the same number-pair except that the sign of the imaginary part is reversed:

$$z^* \equiv (x + iy)^* \equiv x - iy. \tag{6}$$

The process of complex conjugation is obviously equivalent to a reflection about the x -axis in Fig. 1.

Both the product and sum of a complex number and its complex conjugate are real. That is, $(z \cdot z^*)$ and $(z + z^*)$ are real.

2f. Polar Representation: Magnitude and Phase. The “magnitude” of a complex number $z = x + iy$ is written $|z|$ and is defined by

$$|z| \equiv \sqrt{|z|^2} \equiv \sqrt{z^*z} = \sqrt{(x - iy)(x + iy)} = \sqrt{x^2 + y^2}. \tag{7}$$

In terms of the complex plane, the *magnitude* of a complex number z is the polar radius to z 's point (x,y) in the complex plane, as illustrated in Fig. 2. The *phase* ϕ of the point z is defined as the polar angle of the point (x,y) :

$$\phi \equiv \tan^{-1}(y/x).$$

Then:

$$x = |z| \cos \phi; \quad y = |z| \sin \phi$$

and hence:

$$z = x + iy = |z|(\cos \phi + i \sin \phi).$$

2g. Euler's Representation. We now use Euler's² formula, $e^{i\phi} = \cos \phi + i \sin \phi$, to put z in the form:³

$$z = |z| e^{i\phi}, \tag{8}$$

²Pronounced “Oiler.”

³See “Taylor's Series for the Expansion of a Function About a Point” (MISN-0-4) for a Taylor series proof of Euler's formula.

where e is the base of the natural logarithms. This neatly separates z into its magnitude $|z|$ and phase ϕ : compare Eq. (8) and Fig. 2. In the polar representation, complex conjugation reverses the sign of the phase. For example,

$$(4 + 3i)^* = (5 e^{i36.9^\circ})^* = 5 e^{-i36.9^\circ}. \quad \text{Help: [S-2]}$$

Here are some other interesting applications of Euler's representation:

$$z_1/z_2 = \frac{|z_1|}{|z_2|} e^{i(\phi_1 - \phi_2)}$$

$$\sqrt{z} = \pm \sqrt{|z|} e^{i\phi/2} \quad \text{Help: [S-3]}$$

$$\sqrt{i} = \pm e^{i\pi/4} = \pm(\cos \pi/4 + i \sin \pi/4) = \pm \frac{1}{\sqrt{2}}(1 + i). \quad \text{Help: [S-4]}$$

▷ Work out the four applications shown above.

3. Poles of Simple Functions

3a. Real Function Silhouettes. Consider the function

$$f(x) = \frac{a^2}{x^2 + a^2}, \tag{9}$$

where a is a constant. If we vary a we get the $f(x)$ curves shown in Fig. 3. We are going to relate the shape of the central bumps in Fig. 3 to what is going on in the complex plane.

▷ Verify the curves in Fig. 3.

3b. Extending a Function Into the Complex Plane. The real function in Eq. (9), $f(x) = a^2/(x^2 + a^2)$, can be extended into the complex plane by substituting z for x : $f(z) = a^2/(z^2 + a^2)$, where $z = x + iy$. The original function, Eq. (9), is then included as the special case $y = 0$.

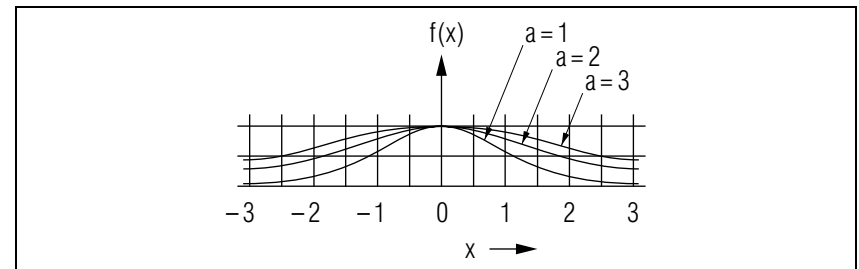


Figure 3. Plot of Eq. (9) for three values of a .

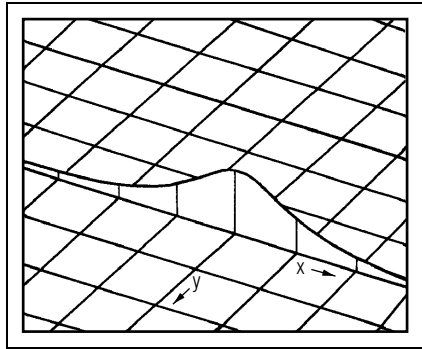


Figure 4. The value of $|1/(z^2+1)|$, plotted along the real axis only.

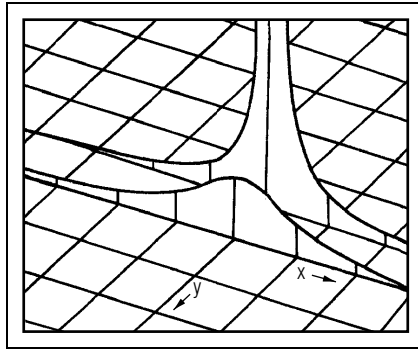


Figure 5. Plot of the function $|1/z^2+1|$, for $y=0$ and $y=-1$.

3c. Plotting a Complex Function's Magnitude. Since $f(z)$ has a single value for each value of x and y , we can plot the magnitude of $f(z)$, $|f(z)|$, as a third dimension extending up from our complex plane. This is shown, for the silhouette along the real axis, in Fig. 4. Note that we have displayed the complex plane in a rather unusual manner, with the y -axis increasing toward the observer. The lower $f(x)$ curve of Fig. 3 is the one shown standing up along the real x -axis in Fig. 4. It is $|f(z)| = |a^2/(z^2+a^2)|$, with $a=1$, plotted for $y=0$.

If we now calculate $f(z)$ with $a=1$ for $y=-1$, we get:

$$|f(x-i)| = \frac{1}{x\sqrt{x^2+4}}; \quad y=-1,$$

shown standing up along $y=-1$ in Fig. 5.

3d. Representing Poles. Notice that $f(z)$ for $y=-1$ becomes infinite as $x \rightarrow 0$. Then $f(z)$ for $a=1$ is said to have a “pole” at the position $x=0, y=-1$.⁴ We can fill in $|f(z)|$ for all points in the complex plane, finding the surface shown in Fig. 6 (where we have plotted $|1/(z^2+1)|$ for all negative values of y). Thus our real axis bump function is seen to be the chopped-off foothill of a mountain that rises to infinity.

3e. Saddle Point and Conjugate Pole. In Fig. 6, notice the line above the imaginary axis as it comes down the front slope of the pole toward you. As that crest line approaches the top of the real axis bump,

⁴Note we have not given a precise definition of poles; see math texts for further discussions.

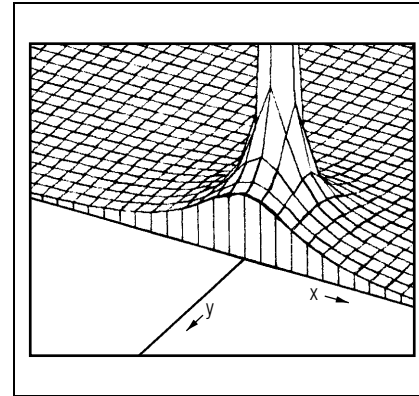


Figure 6. As in Fig. 5, but now plotted for all $y \leq 0$.

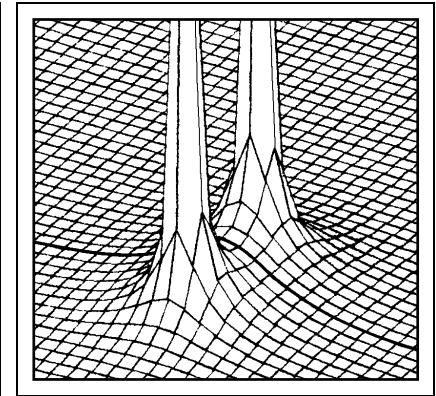


Figure 7. Plot of $|a^2/(z^2+a^2)|$ for $a=1$.

it levels off and becomes, just at the real axis, horizontal. Why does it do that? To see why, factor the denominator of $f(z)$ for $a=1$:

$$\frac{1}{z^2+1} = \frac{1}{(z+i) \cdot (z-i)}.$$

There are two poles, then, at $z_p = \pm i$. These are illustrated in Fig. 7, which also shows the surface above the positive part of the imaginary axis as well as above the negative part. The real and imaginary axes are hidden from sight beneath the surface, but the original real-axis bump curve can still be seen going through the pass between the two mountains.

4. Pole Trajectories

As the parameters of a function change, the poles of the function move around in the complex plane. By way of illustration, consider the function $f(z) = a^2/(z^2+a^2)$, factored to show the poles (the zeros of the denominator):

$$f(z) = \frac{a^2}{(z+ia)(z-ia)}.$$

The poles of this function are obviously at $z_p = \pm ia$. Thus, as the parameter a is either increased or decreased from zero, the two poles move away from the real axis as in Fig. 8. The lines traced out by such movements of the pole positions are called the pole “trajectories.”

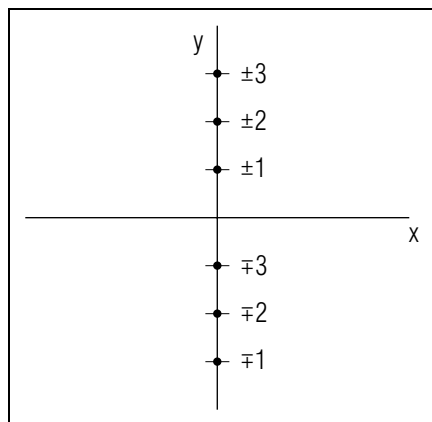


Figure 8. The pole positions of $a^2/(z^2 + a^2)$ for $a = 1, 2, 3$

Half-plane views of pole movements are shown in the Appendix.

Acknowledgments

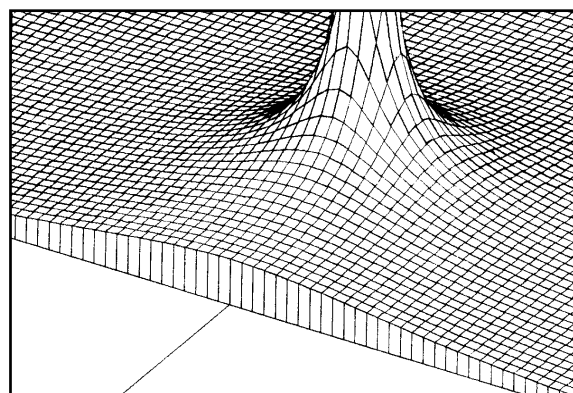
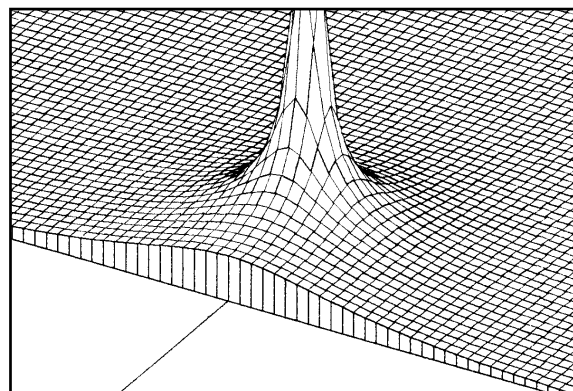
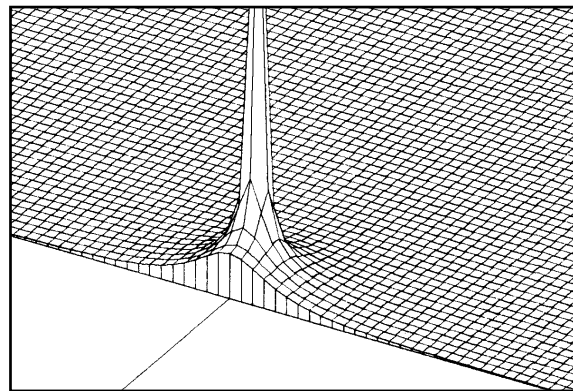
The author wishes to thank The National Science Foundation and IBM for supporting the construction of this module.

Appendix

Half-plane views, corresponding to Fig. 7, are shown in Fig. 9 for $a = 1, 2, 3$. The real-axis bumps in these figures are just those of Fig. 3. Note in Fig. 8 that, as a increases from $-\infty$, one pole moves up the imaginary axis (shown in Fig. 9) while the other pole moves down it (not shown in Fig. 9).

▷ We suggest you use felt-tip markers to color Figures 4-9. That will enhance the three-dimensional appearance of the figures. We suggest you color the vertical faces along the real axis light blue and the line on or above the entire imaginary axis (and up the $y = 0$ line on the vertical face) red.

Figure 9. As in Fig. 8, $a = 1, 2, 3$ (top frame to bottom).



PROBLEM SUPPLEMENT

1. Write down the complex conjugate, the magnitude, and the phase of $z = 8 + 6i$.

2. Locate the poles of

$$f(z) = \frac{25}{z^2 + 8z + 25}$$

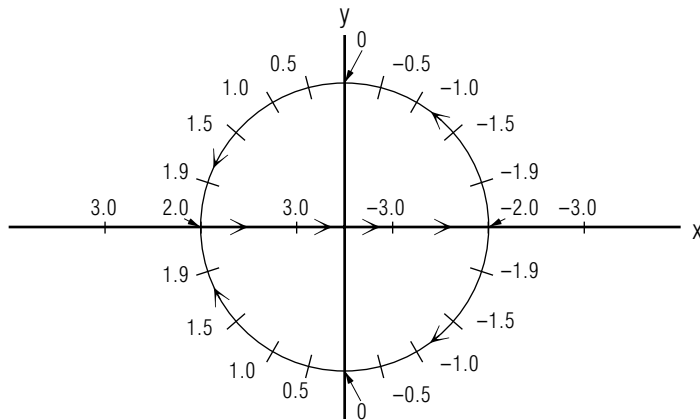
3. Sketch the pole trajectories of

$$f(z) = \frac{1}{z^2 + az + 1}$$

where a is continuously and smoothly increased from $-\infty$ to $+\infty$. Describe the motions of the pole(s) along trajectories.

Brief Answers:

- $z^* = 8 - 6i$; $|z| = 10$; $\phi = \tan^{-1}(3/4) = 36.9$.
- $z_1 = -4 + 3i$; $z_2 = -4 - 3i$. [S-5]
- The arrowheads show the directions the poles travel as a increases from $-\infty$ to $+\infty$. The numbers show the values of a . For clarity, parts of the trajectories are shown slightly displaced from the x -axis. Those parts actually coincide with the x -axis.



SPECIAL ASSISTANCE SUPPLEMENT

S-1 (from TX-2d)

Addition/subtraction rule resulting from $i = \sqrt{-1}$:

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2).$$

Rearranging and factoring,

$$z_1 \pm z_2 = (x_1 + x_2) \pm i(y_1 + y_2), \text{ which is the rule.}$$

Multiplication rule resulting from $i = \sqrt{-1}$:

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 + (i)^2y_1y_2.$$

Using Eq. (2), rearranging, and factoring,

which is the rule.

This can also be written:

$$z_1 \cdot z_2 = |z_1|e^{i\phi_1} \cdot |z_2|e^{i\phi_2} = |z_1| \cdot |z_2|e^{i(\phi_1+\phi_2)}.$$

S-2 (from TX-4)

Computation of Polar Coordinates for the Example

By use of Eqs. (6), (7), and (8):

$$4 + 3i = \sqrt{(4)^2 + (3)^2} e^{i \tan^{-1}(3/4)} = 5e^{i36.9^\circ}.$$

To obtain the complex conjugate, just replace i by $-i$ throughout.

S-3 (from TX-4)

$$\sqrt{z} = (|z|e^{i\phi})^{1/2} = (|z|)^{1/2} (e^{i\phi})^{1/2} = \pm \sqrt{|z|} e^{i\phi/2}.$$

S-4 (from TX-4)

$i = 0 + i = \cos \pi/2 + i \sin \pi/2 = e^{i\pi/2}$. Then:

$$i^{1/2} = (e^{i\pi/2})^{1/2} = \pm e^{i\pi/4} = \pm(\cos \pi/4 + i \sin \pi/4) = \pm \frac{1}{\sqrt{2}}(1 + i).$$

Check the answer by squaring it!

S-5

(from PS-Problem 2)

We want to make the function zero so its inverse will be infinite; the zeros of the function are the pole positions of its inverse. We denote those values of z , that make the function zero, by z_p :

$$az_p^2 + bz_p + c = 0.$$

This is a quadratic expression so we solve it by using the quadratic root formula, which you should have on instant recall:

$$z_p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We check our answer by multiplying out the resulting function to make sure it reproduces the original polynomial :

$$(z - z_1)(z - z_2) = [z - (-4 + 3i)][z - (-4 - 3i)] = \dots$$

Here we denoted the two values of z_p by z_1 and z_2 and we showed the polynomial in factored form (note that either $z = z_1$ or $z = z_2$ makes the function zero).

Apply the quadratic root formula and then do the check yourself!

[S-6]

from TX-3e: Connection between Figures 6 and 7.

