PHYSNET

MISN-0-507

DIELECTRICS - BOUNDARY VALUE PROBLEMS

Electricity and Magnetism

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

DIELECTRICS - BOUNDARY VALUE PROBLEMS

by

R. D. Young

Title: **Dielectrics - Boundary Value Problems**

Author: R. D. Young, Dept. of Physics, Ill. State Univ.

Version: 2/1/2000 Evaluation: Stage B0

Length: 2 hr; 13 pages

Input Skills:

1. Vocabulary: dielectric, dielectric constant, permittivity, electric susceptibility, linear isotropic dielectric, polarization, electric displacement vector, external charge, polarization charge (MISN-0-506); Poisson's equation (MISN-0-505).
2. Express the solution to Laplace's equation in terms of zonal harmonics and cylindrical harmonics (MISN-0-505).
3. State Gauss's law for the electric displacement (MISN-0-506).

Output Skills (Knowledge):

K1. State the boundary conditions for the electric field and the displacement vector at an interface between two dielectric media.

K2. State the forms of Poisson's and Laplace's equations for fields in the presence of dielectric material.

Output Skills (Problem Solving):

S1. Given a simple geometrical arrangement of two dielectric media or a dielectric medium in conjunction with conducting surfaces, use the boundary value conditions to determine the potential, electric field, displacement vector and charge densities in the media.

External Resources (Required):

1. J. Reitz, F. Milford and R. Christy, *Foundations of Electromagnetic Theory*, 4th Edition, Addison-Wesley (1993).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

DIELECTRICS - BOUNDARY VALUE PROBLEMS

by

R. D. Young

1. Introduction

In general, two or more dielectric media are present in a given situation. In fact, the vacuum can always be considered as a dielectric with permittivity ϵ_0. Or there may be at least one dielectric medium and one conducting medium in a problem. In these more complicated problems, the behavior of the field vectors $\vec{E}$ and $\vec{D}$ across the interface between two media is required for a solution. The boundary conditions across such an interface are derived in terms of the tangential component of the electric field $\vec{E}$ and the normal component of the displacement $\vec{D}$. Poisson's equation in the presence of a dielectric medium is derived and specialized to Laplace's equation in the absence of any free charge.

Thus, an electrostatic problem involving linear, isotropic, and homogeneous dielectrics reduces to finding solutions of Laplace's equation in each medium, and joining the solutions in the various media by means of the boundary conditions alluded to above.

2. Procedures

1. Read Secs. 4-7 to 4-8 of the text.

2. Write down or mark in the text the condition on the tangential component of the electric field $\vec{E}$ (eq. 4-42b) and the normal component of the displacement vector $\vec{D}$ (eq. 4-41b) at an interface between two media. Also, write down a sentence or two explaining each of the equations (4-41b) and (4-42b). Be prepared to write down both the equations and the explanatory sentences on the Unit Test.

3. Write down or mark in the text Poisson's and Laplace's equations (4-48 and 4-49, respectively) in the presence of dielectric material.

4. Read Example 4-2, "Dielectric sphere in a uniform electric field," very carefully. This is a prototype example of solving Laplace's equation when a dielectric medium is present using boundary conditions on $\vec{E}$ and $\vec{D}$.

5. Read the *Supplementary Notes* for other examples of solving electrostatic problems involving linear, isotropic, and homogeneous dielectrics. The problems solved in the Notes are numbers 4-7, 4-16, and 4-17 of the text.

6. Solve the following problems:

   4-8, 4-10, 4-15, 4-17

3. Supplementary Notes

1. Problem 4-7

   Given:

   Two dielectric media with dielectric constants K_1 and K_2. Media separated by plane interface. No free charge on interface.

   The angles θ_1 and θ_2 are the angles that the displacement vector makes with a normal to the interface in medium 1 and 2, respectively (see Fig. 1).

   Find - Relationship between θ_1, θ_2 and K_1, K_2.

   Since there is no free charge,

$$D_{1n} = D_{2n} \quad (1)$$

$$E_{2t} = E_{1t} \quad (2)$$

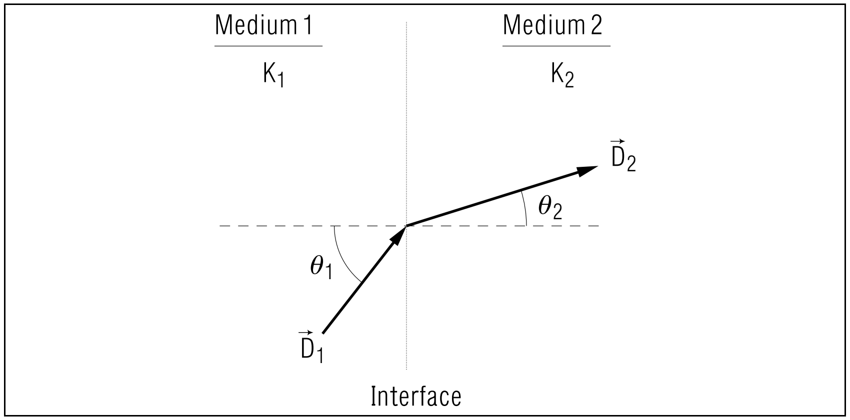

Figure 1. .

where the subscripts n and t mean normal and tangential, respectively. Thus,

$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

$$\frac{K_1}{K_2} = \frac{\tan\theta_1}{\tan\theta_2}$$

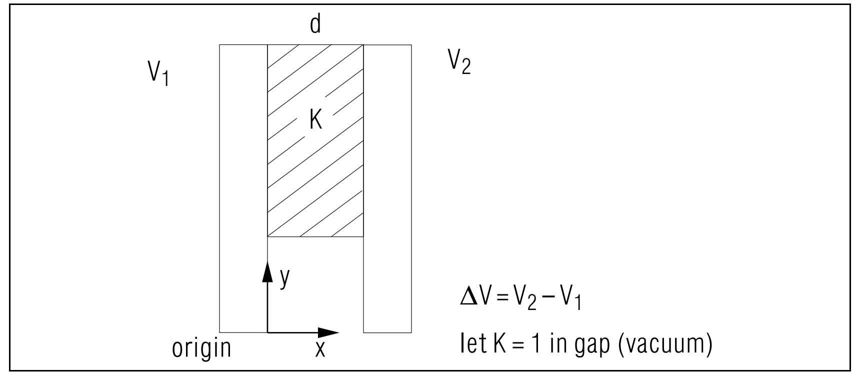

Figure 2. .

2. Problem 4-16

   Given:

   Two parallel conducting plates - separated by d and maintained at potential difference ΔV.

   Slab (dielectric constant K) and uniform thickness d between plates. Slab does *not* completely fill volume between plates.

   Find -

   a) Electric field in dielectric
   b) Electric field in vacuum
   c) Charge density σ on plate in contact with dielectric
   d) Charge density σ on plate in contact with vacuum
   e) Bound charge density σ_p on surface of dielectric

   Solution -

   In dielectric,

$$V = ax + b$$

   In vacuum,

$$V = cx + \ell$$

   But,

$$V(x=0) = V_1 \text{ and } V(x=d) = V_2$$

   Then,

$$V(x=0) = V_1 = b = \ell$$

Also,

$$V(x=d) = V_2 = ad + V_1 = cd + V_1$$

Thus,

$$a = \frac{V_2 - V_1}{d} = \frac{\Delta V}{d}$$

Hence,

$$V(x) = \frac{\Delta V}{d}x + V_1$$

In vacuum and dielectric. Now,

$$E(x) = -\frac{\partial V}{\partial x} = -\frac{\Delta V}{d}$$

in vacuum and dielectric. Inside the conducting plates, $D = E = 0$. So, the boundary condition on the displacement vector (which has only a normal component) is

$$D = \sigma$$

Thus,

$$\sigma = D = K\epsilon_0 E = -K\epsilon_0 \frac{\Delta V}{d}$$

on plate in contact with dielectric.

Thus,

$$\sigma = D = \epsilon_0 E = -\epsilon_0 \frac{\Delta V}{d}$$

on plate in contact with vacuum.

From $\sigma_p = \vec{P} \cdot \hat{n}$,

$$\sigma_p = P = D - \epsilon_0 E = K\epsilon_0 E - \epsilon_0 E = (K-1)\epsilon_0 E$$

So,

$$\sigma_p = -(K-1)\epsilon_0 \frac{\Delta V}{d}$$

3. Problem 4-17

Given:

Conducting sphere (radius R, free charge Q). Sphere floats ha1f-submerged in a liquid dielectric of permittivity E_1. Region above sphere is gas of permittivity E_2.

Find - Electric field in dielectrics.

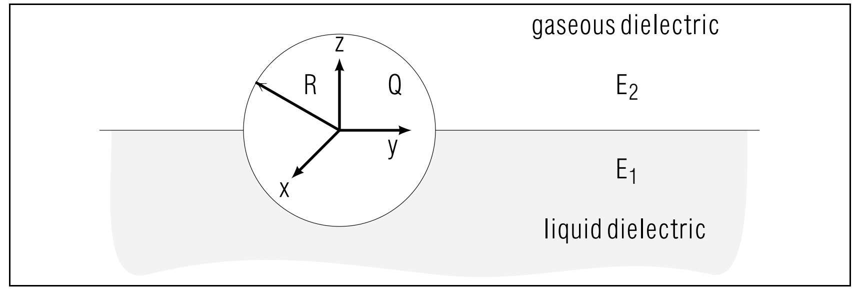

Figure 3. .

Solution -

Select a coordinate system with origin at the center of the sphere, the z-axis perpendicular to the surface of the liquid, and the x-y plane in the surface of the liquid. Then, the problem should be symmetric about the z-axis. Thus, the potential in medium 1 and 2 should be expanded in terms of zonal harmonics. So,

Medium 1

$$V_1 = A_1 + C_1 r^{-1} + A_2 r\cos\theta + C_2 r^{-2}\cos\theta +$$
$$\frac{1}{2}A_3 r^2(3\cos^2\theta - 1) + \frac{1}{2}C_3 r^{-3}(3\cos^2\theta - 1) + \ldots$$

where

$$R < r < \infty, \frac{\pi}{2} \le \theta \le \pi$$

Medium 2

$$V_2 = A_1' + C_1' r^{-1} + A_2' r\cos\theta + C_2' r^{-2}\cos\theta +$$
$$\frac{1}{2}A_3' r^2(3\cos^2\theta - 1) + \frac{1}{2}C_3' r^{-3}(3\cos^2\theta - 1) + \ldots$$

where

$$R < r < \infty, \frac{\pi}{2} \le \theta \le \pi$$

Let $r \to \infty$. Then, V_1 and V_2 must behave like a point charge potential. So,

$$A_2 = A_3 = \ldots = A_2' = A_3' = \ldots = 0$$

Likewise, as $r \to R$, V_1 and V_2 must become constant independent of θ. So,

$$C_2 = C_3 = \ldots = C_2' = C_3' = \ldots = 0$$

Hence,

$$V_1 = A_1 + C_1 r^{-1} \text{ and } V_2 = A_2' + C_1' r^{-1}$$

Thus, $\vec{E}$ is a radial field. Since $\vec{E}$ is independent of θ and continuous at the interface,

$$\vec{E}_1 = \vec{E}_2 = E\frac{\vec{r}}{r}$$

The displacement vector D is parallel to E so D is radial. Apply Gauss' Law to an (imaginary) spherical surface of radius r, concentric with the conducting sphere. Then,

$$\oint \vec{D} \cdot \hat{n} dS = Q = D_1 \frac{A}{2} + D_2 \frac{A}{2}$$

where $A = 4\pi r^2$. So,

$$Q = \epsilon_1 E \cdot 2\pi r^2 + \epsilon_2 E \cdot 2\pi r^2$$

$$E = \frac{Q}{2\pi(\epsilon_1 + \epsilon_2) r^2}$$

and

$$\vec{E} = \frac{Q}{2\pi(\epsilon_1 + \epsilon_2) r^3} \vec{r}$$

You can calculate the charge densities.

Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.