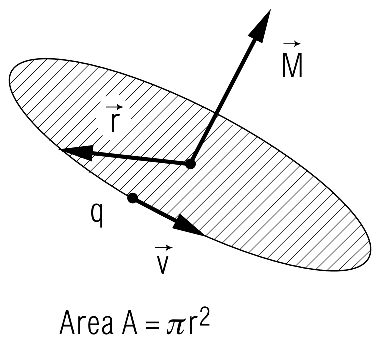
MISN-0-251

QUANTIZED ANGULAR MOMENTUM



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

QUANTIZED ANGULAR MOMENTUM

by

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Title: **Quantized Angular Momentum**

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Input Skills:

1. Calculate the angular momentum of a particle in a circular orbit (MISN-0-41).
2. Helpful: Calculate the magnetic dipole moment of a current loop and the energy of a dipole in a magnetic field (MISN-0-130).

Output Skills (Knowledge):

K1. Derive the expression which relates the magnetic moment of a circulating charged particle (in a circular orbit) to the angular momentum of that particle.

K2. Calculate the possible energy change that occurs when a circulating charged particle is placed in an external magnetic field. Compare the classical result with the correct quantum mechanical result.

Output Skills (Rule Application):

R1. Given a system which has two component parts, which have angular momentum quantum numbers ℓ_1 and ℓ_2, determine the possible values of the total angular momentum of the system and the possible values of its projection along any given direction.

External Resources (Required):

1. *Physics*, M. Alonso and E. J. Finn, Addison-Wesley Publ. Co., Reading, MA (1970). For access to these readings, see this module's *Local Guide*.
2. *Elementary Modern Physics*, R. T. Weidner and R. L. Sells, Allyn and Bacon, Boston (1980). For access, see this module's *Local Guide*.

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OF PROJECT PHYSNET

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1. Introduction

The electromagnetic interaction (a single basic interaction, not two independent ones) is the fundamental interaction responsible for all the normally observed properties of matter: chemical, biological, and physical properties, etc. This statement, however, requires qualification. These properties follow from the electromagnetic interaction only after you accept the existence of the nucleus in the nuclear model of the atom and some basic changes in the laws of physics. (Properties of the nucleus require the recognition of a new interaction in addition to the electromagnetic.) These basic changes in the physical laws come to be recognized only when a detailed study of atomic systems is made.

The observed details of atom structure are not explained by the so-called classical laws of physics. The introduction of quantization opened the way for the explanation of these phenomena, leading to the development of quantum mechanics. In particular, the space quantization of orbital and spin angular momentum combined with the Pauli Exclusion Principle explains some of the observed features of the electromagnetic structure of atoms, molecules and solids. Angular momentum, specifically, is the subject of this module.

2. Suggested Readings

a. Read section 18.5 of AF[1] and, if you wish, sections 7-4 and 7-5 of WSM.[2]

b. Read sections 7.6, 7.7 of WSM.

[1] AF is *Physics*, M. Alonso and E. J. Finn, Addison-Wesley Publ. Co., Reading, MA (1970). For access to these readings, see this module's *Local Guide*.

[2] WSM is *Elementary Modern Physics*, R. T. Weidner and R. L. Sells, Allyn and Bacon, Boston (1980). For access to these readings, see this module's *Local Guide*.

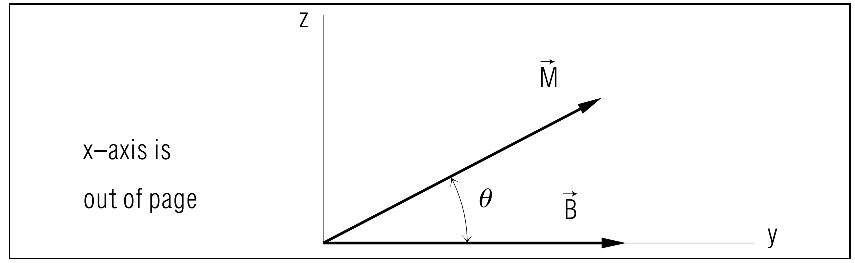

Figure 1. Illustration of magnetic field and magnetic moment vectors. The x-axis is out of the page.

3. Comments on the Readings

In AF, Fig. 18.8 (a), above the words "magnetic field," AF omitted a smeared-out continuous spectrum between $+(e/2m)BL$ and $-(e/2m)BL$. This spectrum is to be contrasted with the $(2\ell+1)$ set of discrete energies of Figure 18.8 (b).

At the bottom of the second column of page 399 of AF you will find the expression for the potential energy of a magnetic dipole $\vec{M}$, in a region where a uniform magnetic field $\vec{B}$ exists: $E_p = -\vec{M}\cdot\vec{B}$. The source of this expression is not mysterious and can easily be derived: see MISN-0-130 for a derivation.

Consider a uniform magnetic field along the y-axis and a magnetic moment $\vec{M}$ in the $y-z$ plane, as shown in Fig. 1. The torque on $\vec{M}$, (again referring to MISN-0-130) tending to line up with the magnetic field $\vec{B}$ is $\vec{\tau} = \vec{M} \times \vec{B} = -MB\sin\theta\,\hat{x}$ (into the page).

To keep it from lining up with $\vec{B}$, an external agent must exert a counter-torque of $+MB\sin\theta\,\hat{x}$ (out of page, hence counterclockwise in the sketch). To rotate infinitesimally through an angle $d\theta$ requires you to do work $|\vec{\tau}|d\theta$ if rotation is about the same axis as $\vec{\tau}$.

The torque on $\vec{M}$, (again refer to Unit 130) tending to line it up with the field $\vec{B}$ is $\tau = \vec{M} \times \vec{B}\sin\theta\hat{x} = -MB\sin\theta\hat{x}$ (into page). To keep it from lining up with $\vec{B}$, an external agent must exert a *countertorque* of $+MB\sin\theta\hat{x}$ (out of page, hence counterclockwise in sketch). To rotate this moment infinitesimally through an angle $d\theta$ requires you to do work $|\vec{\tau}|d\theta$ if rotation is about same axis as $\vec{\tau}$.

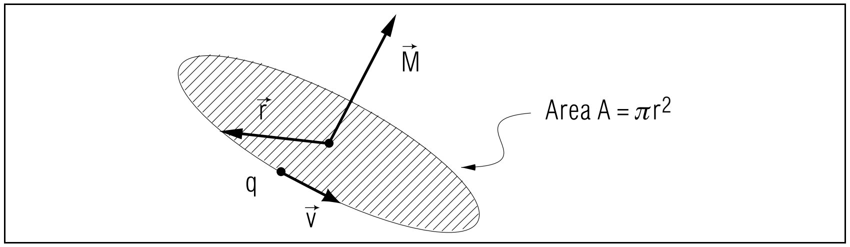

Figure 2. A charged particle in a circular orbit.

Note that $|\vec{\tau}|d\theta$ is the angular analog of $|\vec{F}|ds$ for the case where displacement ds is along the direction of $\vec{F}$. So the work needed to rotate $\vec{M}$ from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} MB \sin\theta \, d\theta$$

$$W = -(MB\cos\theta_2 - MB\cos\theta_1)$$

$$W = E_p(\theta_2) - E_p(\theta_1),$$

from the definition of potential energy. So the potential energy function is

$$E_p(\theta) = -MB\cos\theta = -\vec{M}\cdot\vec{B}$$

Note carefully, that not only is the *magnitude* of the angular momentum vector $\vec{L}$ quantized (in the correct quantum theory): $L = \sqrt{\ell(\ell+l)}\hbar$ where $\ell = 0, 1, 2, \ldots \infty$, any positive integer or zero, but also its possible projections along any arbitrary direction are limited to values $m_\ell\hbar$ ("space quantization") where m_ℓ has possible values from $+\ell$ to $-\ell$ with all integer steps in between. The paragraph following equation 18.21 of AF and the discussion on the Zeeman effect cover this. However, there they refer to the arbitrary direction as the "z-direction." This you may find misleading. The z-direction can be *any* direction in space: the possible projections would be the same. For example, a direction determined by the orientation of a magnetic field $\vec{B}$ may be such a direction.

Question: What are the possible projections of $\vec{L}$ of an atom along the direction of $\vec{B}$ if the atom is in a state characterized by $\ell = 2$? The answer is $+2\hbar$, $\hbar$, 0, $-\hbar$ and $-2\hbar$. The projection of $\vec{L}$ on any axis has only those five possible values.

Question: How is the magnetic (dipole) moment of a circulating charged particle, like the one in Fig. 2, related to its angular momentum?

The magnetic moment of a current loop is defined as $|\vec{M}| = IA$ with the direction of $\vec{M}$ determined by the right-hand rule (perpendicular to the plane of the circular orbit, in the direction of the thumb of the right hand when the fingers encircle the orbit in the direction of the current). What is the current associated with this circulating charge? Suppose it goes around ν times per second; that's the frequency of the orbiting particle. In radians per second $\omega = 2\pi\nu$ which is related to the speed v in the orbit by $v = \omega r$. The current associated with this charge is the number of coulombs per second that go past any point in the orbit. That is $q\nu$, the charge times the number of times it goes by per second. Thus

$$|\vec{M}| = IA = (q\nu)(\pi r^2) = \left(\frac{qv}{2\pi r}\right)(\pi r^2)$$

But from the definition of angular momentum,

$$|\vec{L}| = mvr \qquad \text{(in same direction as } \vec{M}\text{)},$$

$$\vec{M} = \left(\frac{q}{2m}\right)\vec{L}$$

If q is negative $\vec{L}$ and $\vec{M}$ are oppositely directed.

How is the energy of a system consisting of a circulating charged particle affected if it is placed in a magnetic field? Consider the system with no magnetic field present. The charged particle has mass and a velocity so it has a kinetic energy in its orbit. It is bound in its orbit by some force (presumably, the Coulomb force) so it has a potential energy. It thus has some total energy E_0. What happens when the $\vec{B}$-field is turned on? There is an additional potential energy that the system gets which is given by $E_B = -\vec{M}\cdot\vec{B}$ so that now the total energy is

$$E_{\text{total}} = E_0 - \vec{M}\cdot\vec{B}$$

So the energy diagram (energy value of system increasing upward on diagram) becomes as shown in Fig. 3.

And because according to the Newtonian theory of mechanics all angles θ between $\vec{M}$ and $\vec{B}$ are possible, the system can have any energy between the maximum and the minimum. The shaded area of the previous figure shows the region of possible energies for the circulating charge system in the field depending upon the angle $\vec{M}$ makes with $\vec{B}$, the energy spread being $\Delta E = 2|\vec{M}||\vec{B}|$.

So if you had a large collection of such circularly orbiting particles each with magnetic moment $\vec{M}$ ("atoms"), their orientations would be

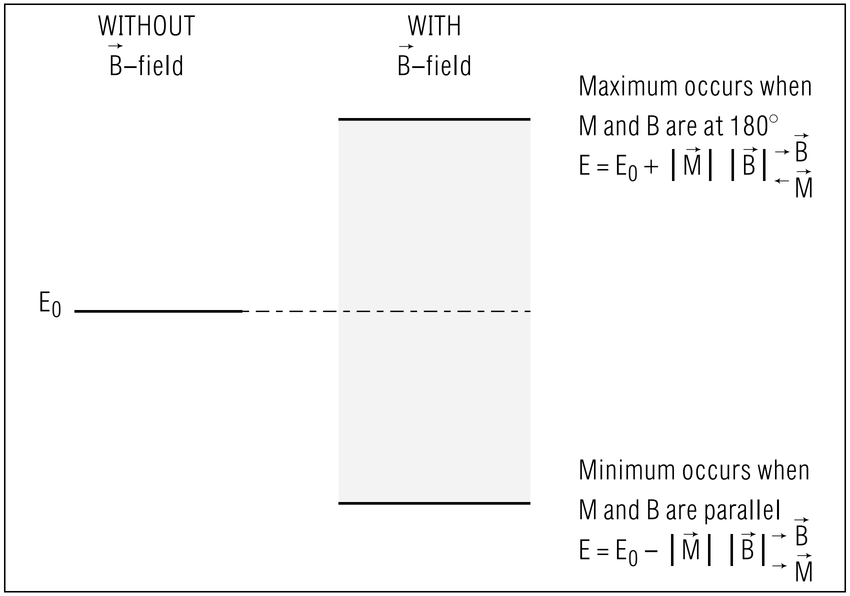

Figure 3. Illustration of a magnetic field spreading an energy level.

random with the moments, $\vec{M}$, pointing in all possible directions. If you then imposed an external magnetic field on this collection and measured the individual energies of these atoms (How do you measure the possible energy states of atoms? See Units 215 and 216), the energies you'd measure, according to the classical theory, would cover all possible energies in this interval $\Delta E = 2|\vec{M}||\vec{B}|$ around E_0. Without the field you'd detect only energy E_0 for every atom. However, what you'd *actually* observe disagrees with the classical theory.

Not all energies between $E_0 + MB$ and $E_0 - MB$ are possible, only certain sharply defined energies. The quantum mechanical explanation for this is that the *possible projections* of the angular momentum $\vec{L}$ along any direction is quantized, only certain multiples of $h/2\pi$ being possible, $\vec{L}$ is also quantized, with only values $[\ell(\ell + l)]^{1/2}(h/2\pi)$ being possible, ℓ being zero or positive integers. The projection of $\vec{L}$ along any given direction, for a given integer ℓ, can only have values between $\ell\hbar$ ($\hbar \equiv h/2\pi$) and $-\ell\hbar$ with all integer values between $+\ell$ and $-\ell$. Hence, instead of the continuous energy spectrum shown above (with spread $2MB$), there are between the maximum and minimum only a discrete set of energies

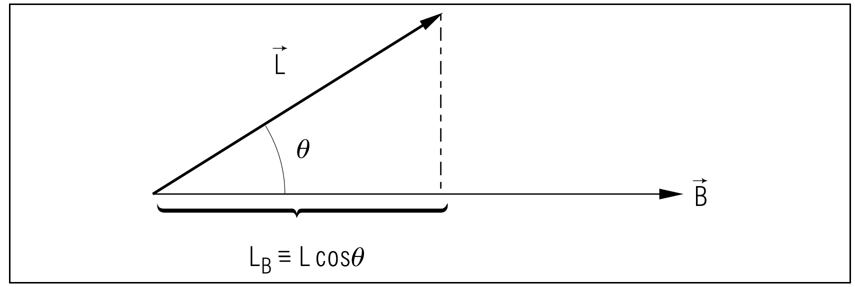

Figure 4. Obtaining the component of $\vec{L}$ along the $\vec{B}$-axis.

possible, determined by the value of ℓ, there being $2\ell+1$ possible energies. That's because $\vec{M} = q\vec{L}/2m_0$ and $\vec{M}\cdot\vec{B} = q\vec{L}\cdot\vec{B}/2m_0 = qLB\cos\theta/2m_0$. But $L\cos\theta = L_B$ is the projection of $\vec{L}$ along the direction determined by $\vec{B}$ (see Fig. 4).

The angular momentum quantization rule restricts L_B to values $\ell\hbar$, $(\ell-1)\hbar$, $(\ell-2)\hbar$, ..., etc. until by successive subtractions of unity from ℓ you get to $-\ell$. Thus

$$\vec{M}\cdot\vec{B} = \left(\frac{q\hbar B}{2m_0}\right) L_B$$

where L_B is quantized (L_B is usually written as $m\hbar$ where m takes on the possible values ℓ, $\ell - l$, $\ell - 2$, ..., $-\ell$. Don't confuse this m with the particle mass, here written as m_0.

4. Adding Angular Momenta

Suppose you have two angular momenta, $\vec{L}_1$ and $\vec{L}_2$. What's the total angular momentum of the combined system for which these two angular momenta are separate components? Classically, you'd just add the two parts vectorially to obtain the resultant $\vec{L}_R$ (see Fig. 5).

For given values for the magnitude of $\vec{L}_1$ and $\vec{L}_2$ the magnitude of $\vec{L}_R$ can have a range of values between the maximum $|\vec{L}_1|+|\vec{L}_2|$ when the two are parallel and $|\,|\vec{L}_1|-|\vec{L}_2|\,|$ when they are oppositely directed, depending upon the angle between the direction of $\vec{L}_1$ and $\vec{L}_2$. This angle can vary continuously from 0°to 180°. That's what is expected to be true from the classical picture. However, because of the quantization of angular momentum each of $\vec{L}_1$, $\vec{L}_2$, and the resultant $\vec{L}_R$ can only have possible magnitudes which are given by $[\ell(\ell+1)]^{1/2}\hbar$ with the ℓ's restricted to

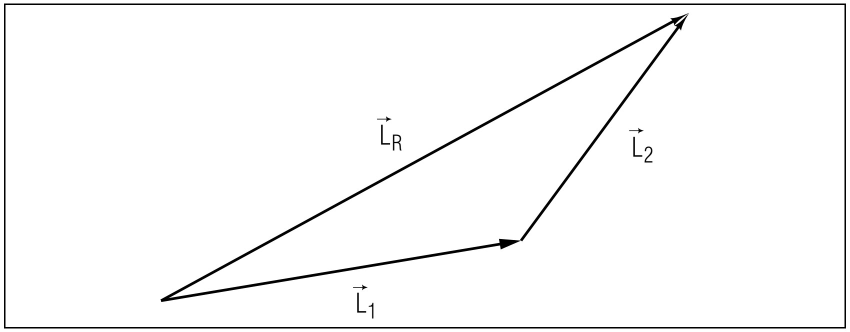

Figure 5. Classical addition of two angular momenta.

integers, and their projections along a given arbitrary direction are each given by some $m\hbar$. If $\vec{L}_1$ has for its projection along some given direction the value $m_1\hbar$, then it's clear that the projection of $\vec{L}_R$ (the resultant) along the same direction must be $(m_1+m_2)\hbar$. If that is the projection of $\vec{L}_R$ what is $|\vec{L}_R|$? It obviously has some value $[\ell(\ell+1)]^{1/2}\hbar$ with an ℓ-value compatible with the value $(m_1+m_2)\hbar$ for its projection. For concreteness consider the following example. Suppose:

$$|\vec{L}_1| = [\ell_1(\ell_1+1)]^{1/2}\hbar \qquad \text{with } \ell_1 = 1 \text{ and}$$
$$|\vec{L}_2| = [\ell_2(\ell_2+1)]^{1/2}\hbar \qquad \text{with } \ell_2 = 2.$$

i. What are the possible values for the projection quantum number, m_1? Ans: 1, 0, and -1.

ii. What are the possible values for the projection quantum number, m_2? Ans: 2, 1, 0, -1, -2.

iii. If along a specified direction the projection m_1 has the value +1 while $m_2 = -2$, what is the projection of the angular momentum $\vec{L}_R$ along that same direction? Ans: $-\hbar$

iv. If this is the value of the projection of $\vec{L}_R$ what is the value of $|\vec{L}_R|$, the magnitude of the resultant angular momentum? Ans: That isn't unique. Many ℓ-values have $m = -1$ as one of its possible projection quantum numbers. In fact, every integer except $\ell = 0$ has $m = -1$ among its possible projections.

v. What possible combinations of m_1 and m_2 can give you an m associated with the resultant equal to $m = 3$? Ans: Only if $m_1 = +1$ and $m_2 = +2$. Write it as (1,2) for convenience. For the given $\ell_1 = 1$ and $\ell_2 = 2$ this is the possible way that the projection of $\vec{L}_R$ can have projection quantum number $m = 3$.

vi. What possible combinations of m_1 and m_2 can give you $m = 4$? Ans: No combination. With $\ell_1 = 1$ and $\ell_2 = 2$ the maximum m_1 is $m_1 = 1$ and the maximum m_2 is $m_2 = 2$, so that the maximum $m = 3$. This then tells you that the maximum ℓ_R is

$$|\vec{L}_R| = [\ell_R(\ell_R+1)]^{1/2}\hbar \text{ is } \ell_R = 3.$$

vii. What possible combinations of m_1 and m_2 give you $m = 0$? Ans: (1,−1), (−1,1), (0,0). Notice that if $m_2 = 2$ or -2 there's no way m can be zero because the maximum and minimum m_1 values are 1 and −1 respectively.

viii. With all possible m_1 values combined with all possible m_2 values list all the possible values m can have (allow repeated values for m if the same m, as in (vii) above, can be obtained by different combinations of m_1 and m_2). Ans: 3, 2, 1, 0, −1, −2, −3, 2, 1, 0, −1, −2, 1, 0, −1. There are 15 different values. Associated with each of ℓ_1 and ℓ_2 there are $(2\ell_1+1)$ and $(2\ell_2+1)$ values of m_1 and m_2. So the total number of combinations is the product $(2\ell_1+1)(2\ell_2+1)$.

ix. What are the possible ℓ_R values that you can associate with these m's? Ans: The 7 numbers $m = 3$, 2, 1, 0, −1, −2, and −3 are clearly the projections associated with $\ell_R = 3$. The 5 numbers 2, 1, 0, −1, −2, belong to $\ell_R = 2$ and the remaining m's equal to 1, 0, −1 belong to $\ell_R = 1$. So concluding: If $\ell_1 = 1$ and $\ell_2 = 2$ the only possible values the resultant $\vec{L}_R$ can have are $[\ell_R(\ell_R+l)]^{1/2}\hbar$ with $\ell_R = 3$, 2, or 1. No other resultant is possible.

5. The General Addition Rule

If two independent angular momenta contribute to a system's total angular momentum,

$$\vec{L} = \vec{L}_1 + \vec{L}_2 \qquad \text{(classical mechanics)},$$

then quantum mechanically the allowed values of $\vec{L}$ and $\vec{L}_z$ are:

$$|\vec{L}| = \sqrt{\ell(\ell+1)}\hbar$$
$$|\vec{L}_z| = m_\ell\hbar$$

where ℓ is any integer in the range

$$|\ell_1 - \ell_2| \le \ell \le (\ell_1 + \ell_2)$$

and, for a given ℓ, m_ℓ is any integer in the range

$$-\ell \le m_\ell \le \ell.$$

The cover of this module illustrates the popular "triangular" method of adding angular momenta, wherein you pretend that ℓ_1 and ℓ_2 are integer-length vectors ($\ell_1 = 2$ and $\ell_2 = 1$ on the cover). This method is not meant to be taken literally.

Acknowledgment

I would like to thank Bill Lane for comments that have improved this module.

LOCAL GUIDE

To obtain the readings for this module, go to the Physics-Astronomy Library and ask for "the readings for CBI unit 251."

PROBLEM SUPPLEMENT

1. Problem 18.9 in AF.
2. Problem 18.10 in AF.
3. Problem 18.11 in AF.
4. On the same sort of plot where the (classical) possible energies of an atom in a magnetic field were sketched above, plot the energy spectrum expected on the basis of quantum mechanics for the cases where $\ell =$ 0, 1, and 4.
5. Suppose you observe the energy spectrum of an atomic state when that atom is in an external magnetic field and you see that the given energy state splits up into 9 closely spaced energy levels when the magnetic field is turned on. What is the value of $|\vec{L}|$ for that state?
6. You make an observation of the projection of the angular momentum of a system along some specified direction and measure that projection to be $3\hbar$. What are the possible values of $|\vec{L}|$ for which this could be the projection?
7. In the same way as in the guided illustration in the text above, consider the addition of two angular momenta with $\ell_1 = 4$ and $\ell_2 = 2$.
   a. How many possible combinations of m_1 and m_2 are there of these two sets?
   b. How many different ways can the resultant m give $m = 0$? $m = 1$? $m = -2$? List them.
   c. List all of the possible values of m.
   d. Collecting these above m-values into suggestive groupings, identify them according to the ℓ-values that they belong to.
   e. Hence, what are the possible values of ℓ_R that you can get by combining $\ell_1 = 4$ and $\ell_2 = 2$?

Brief Answers:

1. $|\vec{L}| = \sqrt{6}\hbar$; $L_z = 2\hbar$, $\hbar$, 0, $-\hbar$ and $-2\hbar$. Angles (similar to Fig. 18.7), there are 5 of them, 2 above 0°, 0°, and 2 below 0°.
2. When $n = 3$, there are $\ell = 2$, $\ell = 1$, $\ell = 0$ as the possible value of ℓ (see Eq. 18. 21) . Assuming system is in the state $n = 3$, $\ell = 2$ then there are 5 levels into which the magnetic field splits the state. If it's state $n = 3$, $\ell = 1$, the field splits it into 3 levels (such as Fig. 18.8 (b) illustrates with a nine-level splitting) and if $\ell = 0$, there is no splitting. The magnetic energy difference (for $\ell = 2$) is $\Delta E = e\hbar B/2m_e = 2.31 \times 10^{-4}$ eV between adjacent levels. (Draw a level diagram similar to Figure 18.8 (b) for this 5-level system). Note that AF has an error in μ_B: $\mu_B = 9.2732 \times 10^{-24}$ J/T $= 5.7883 \times 10^{-5}$ eV/T.
3. Again use Eq. 18.21, $\ell = n - 1$ for circular orbit.
4. See Figure 18.8(b) on page 400 of AF. If $\ell = 0$ there is no shift, E_0 stays E_0, a single line. With $\ell = 1$ there are 3 lines evenly spaced, 9 with $\ell = 4$.
5. $|\vec{L}| = (20)^{1/2}\hbar$ because there are $(2\ell+1)$ levels which means that $\ell = 4$.
6. If $m\hbar = 3\hbar$, the associated ℓ may be 3, 4, 5, ... up to ∞ (any integer greater than 2). Hence $|\vec{L}|$ can have possible values $(12)^{1/2}\hbar$, $(20)^{1/2}\hbar$, $(30)^{1/2}\hbar$, ..., etc.
7. a. 45
   b. $m = 0$ can occur 5 different ways: (2,−2), (1,−1), (0,0), (−1,1), and (−2,2). $m = 1$ can occur 5 different ways: (3,−2), (2,−1), (1,0), (0,1) and (−1,2). $m = -2$ also can occur 5 different ways: (0,−2), $(-1, -1)$, $(-2, 0)$, $(-3, 1)$, and $(-4, 2)$.
   c. and d: 6, 5, 4, 3, 2, 1, 0, −1, −2, −3, −4, −5, −6; 5, 4, 3, 2, 1, 0, −1, −2, −3, −4, −5; 4, 3, 2, 1, 0, −1, −2, −3, −4; 3, 2, 1, 0, −1, −2, −3; 2, 1, 0, −1, −2.
   e. ℓ_R can be 6, 5, 4, 3, 2. Note that if you add up the $(2\ell_R + 1)$ for each of these ℓ_R values you get $13 + 11 + 9 + 7 + 5 = 45$ which is exactly the number of possible ways of combining m_1 and m_2: $(2\ell_1 + 1)(2\ell_2 + 1)$.

MODEL EXAM

1. Starting from the definition of the magnetic moment of a point charge moving in a circular orbit, derive the relation which gives the magnetic moment of an electron which is in a circular orbit of angular momentum $\vec{L}$. Get $\vec{M} = e/2m_e\vec{L}$.

2. Using this result derive the expression for the possible energy values that an electron has in a state of orbital angular momentum $\vec{L}$ with definite orbital quantum number ℓ when the atom is in a uniform magnetic field $\vec{B} = B_0\hat{x}$.

   Get: $E = E_0 + e\hbar B_0 m_\ell/(em_e)$.

   How many possible energy states are there? What is E_0? What is the spacing of the levels? How many different energy states are there when $\vec{B} \to 0$?

3. One Bohr magneton equals 5.66×10^{-4} electron-volts per tesla. In a field $B_0 = 1.7\,\text{teslas}$, what is the energy gap between adjacent energy levels in a state of an atom for which $\ell = 6$? What is the total energy gap between the highest and the lowest energy levels? $1\,\text{tesla} = 1\,\text{weber/m}^2$.

4. Suppose a system has two contributing angular momenta, $\ell_1 = 1$ and $\ell_2 = 2$. State all possible values of the total angular momentum quantum numbers $(\ell,\, m_\ell)$.

Brief Answers:

1. See page 400 of AF, shaded region.

2. See text (there are $2\ell + 1$ levels).

3. Spacing between adjacent levels is $\mu_B B_0$, over-all energy "band" width is $2\mu_B B_0 \ell$ [see Figure 18.8(b)].

4. $(\ell = 3;\ m_\ell = -3, -2, -1, 0, 1, 2, 3)$
   $(\ell = 2;\ m_\ell = -2, -1, 0, 1, 2)$
   $(\ell = 1;\ m_\ell = -1, 0, 1)$