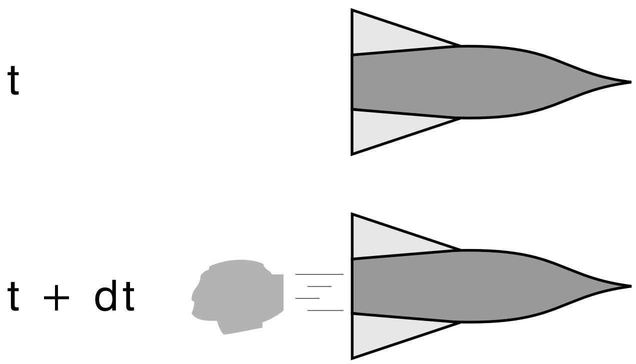
PHYSNET

MISN-0-19

MASS CHANGING WITH TIME: THE VERTICAL ROCKET, ETC.



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MASS CHANGING WITH TIME: THE VERTICAL ROCKET, ETC.

by

Peter Signell

Title: **Mass Changing With Time: The Vertical Rocket, Etc.**

Author: P. Signell, Michigan State University

Version: 1/7/2002 Evaluation: Stage 0

Length: 1 hr; 16 pages

Input Skills:

1. Differentiate logarithms, understand differentials, integrate reciproca1s (MISN-0-1).
2. Vocabulary: frame of reference, relative velocity (MISN-0-11).
3. Apply the law of conservation of momentum to collisions in one dimension (MISN-0-14).
4. Apply Newton's Second Law to one-dimensional constant-force problems (MISN-0-16) .

Output Skills (Knowledge):

K1. Neglecting air resistance derive the velocity of a vertically rising rocket, which has a high constant burn rate, as a function of time and as a function of fuel consumption.

K2. Show that gravity can be neglected for the above case during the burn.

K3. Show that two rocket stages are more efficient than one.

Output Skills (Problem Solving):

S2. Show that the equations found in Kl agree qualitatively and within given limits with what one would expect.

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

MASS CHANGING WITH TIME: THE VERTICAL ROCKET, ETC.

by

Peter Signell

1. Introduction

Ordinary locomotion on earth relies heavily on friction - the interaction of a vehicle, be it bicycle, car, ship, or aircraft, with its environment. For a rocket outside the earth's atmosphere there is neither earth nor air nor water to push against; the rocket must achieve its acceleration another way. It is via fuel mass expulsion that such propulsion is achieved and controlled. Here we investigate the propulsion of the vertical rocket; that is, of a rocket propelled vertically upward against a uniform gravitational field. The commonplace use of rockets for launching communications satellites and for interplanetary explorations makes this subject one of considerable technological interest.

The vertical rocket is a special case of the more general class of problems involving systems of variable mass, due here to the expulsion of fuel. The solution of other variable mass problems proceeds in exactly the same manner as for the vertical rocket, except that the external force exerted on the system is not always that of gravity. One example is that of a jet aircraft, where the external drag force comes mainly from air resistance.

2. The Gravity-Free Rocket Equation

2a. Rocket Momentum Balances Exhaust Momentum. We will deal first with a rocket in outer space where gravitational forces are weak and can be neglected. How can we explain the ability of this rocket to accelerate? Our rocket can be considered to consist of two parts; the fuel and the shell-payload. Expansion of the fuel during burning results in its continuous expulsion rearward in the'form of burned-out exhaust gases. Since there is no external force on the total system, the system's momentum is conserved.[1] This implies that any rearward momentum acquired by the fuel as exhaust must be balanced by a newly-acquired momentum of the rocket in the forward direction. Furthermore, the center-of-mass

[1]See "Momentum: Conservation and Transfer" (MISN-0-15).

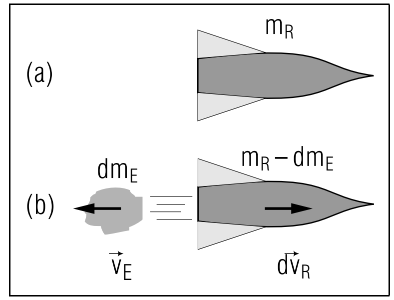


Figure 1. A rocket is shown: (a) at one instant; (b) at a time dt later.

of the entire system is left undisturbed as the exhaust gases and rocket move away from it in opposite directions.

Another way of describing the rocket's acceleration is in terms of Newton's third law.[2] The rearward acceleration of the exhaust implies a rearward force on it by the rocket but this in turn implies the existence of an equal but opposite forward force-exerted on the rocket by the exhaust gases.

2b. Thrust as a Function of Exhaust Velocity, Burn Rate. We can use conservation of momentum for the gravity-free ($g = 0$) rocket to derive its acceleration as a function of its design parameters. This derivation is easiest to make if we imagine ourselves as moving along parallel to the rocket with a constant velocity which coincides with the rocket's at a specified instant; then during the succeeding time increment, we observe the momentum changes in the various parts of the system. In Figure la we see the rocket with mass mR at the instant when it is apparently at rest because we are moving along with it.

In Figure lb we see it at a time dt later when it has expelled a mass of exhaust gas dm_E rearward with velocity $\vec{v}_E$ relative to the rocket. The fuel burn rate R connects dm_E to the time increment dt:

$$dm_E = R\,dt\,. \tag{1}$$

We now use conservation of momentum for our isolated system to determine the increment of velocity, $d\vec{v}_R$, acquired by the rocket:[3]

$$0 = d\vec{p} = d\vec{p}_R + d\vec{p}_E = m_R d\vec{v}_R + \vec{v}_E dm_E\,, \tag{2}$$

[2]See "Particle Dynamics-The Laws of Motion" (MISN-0-14).
[3]The momentum form of Newton's Second Law, $F = d(mv)/dt$, can be used for rocket problems only with great care. We advise against it.

where we have (properly) neglected a product of infinitesimals. There are several connections we can make between the quantities in Eq. (2):

$$dm_E = -dm_R$$
$$\vec{v}_E \equiv v_E \hat{v}_E = -v_E \hat{v}_R\,; \qquad (3)$$
$$d\vec{v}_R = dv_R \hat{v}_R\,.$$

Combining Eqs. (1), (2), and (3), we find the rocket's acceleration:

$$a_R = \frac{v_E R}{m_R}\,, \qquad (4)$$

and hence the force on it produced by the exhaust gases:

$$F_R = v_E R\,. \qquad (5)$$

This force is called the rocket engine's thrust and its linear dependence on v_E and R checks with what one would expect intuitively. Although we have derived Eqs. (4) and (5) for the case of an observer moving at a certain constant velocity, the same force and acceleration values will be found by a stationary observer.[4] Thus Eqs. (4) and (5) are quite general.

2c. Velocity as a Function of Rocket Parameters. The equation relating a rocket's velocity to its fuel expenditure is particularly simple and obviously has practical application. To derive it we integrate the differential form of Eq. (4),

$$dv_R = -\frac{v_E}{m_R} dm_R\,,$$

to obtain the change in velocity:

$$\Delta v_R = v_E\ \ell n \left(\frac{m_{Ri}}{m_{Rf}}\right)\,. \qquad (6)$$

Here m_{Ri} is the initial rocket mass (at ignition) and m_{Rf} is its final mass (at burn-out).

Assuming that the initial rocket mass consists of fuel mass m_F and shell-payload mass m_S, and that the final rocket mass consists only of the shell-payload, we obtain the velocity change in terms of the fuel expenditure:

$$\Delta v_R = v_E\ \ell n \left(1 + \frac{m_F}{m_S}\right)\,. \qquad (7)$$

All parts of this equation check with what one would expect intuitively.

[4]This is because accelerations add vectorially and the acceleration of one constant-velocity observer with respect to another (of differing velocity) is zero. For further details see "Relative Linear Motion, Frames of Reference" (MISN-0-11).

2d. Velocity as a Function of Time. Equation (6), for the change in velocity as a function of mass, can be written as an implicit function of time:

$$v_R(t) = v_R(t_i) + v_E\ \ell n \left(\frac{m_R(t_i)}{m_R(t)}\right);\ t_i \le t \le t_f\,, \qquad (8)$$

where t_i is the ignition time and t is any succeeding time up to the shut-down time t_f. The time dependence can be made explicitly if we specify $m_R(t)$. For example, a constant burn rate

$$R = -dm_R/dt = \text{ positive constant} \qquad (9)$$

will produce (upon integration) a linear decrease of the rocket mass with time:

$$m_R(t) = m_R(t_i) - (t - t_i)R;\ t_i \le t \le t_f\,,$$

where t_f is easily found from:

$$\Delta t = \frac{\Delta m_R}{R};\ \text{where } \Delta t \equiv t_f - t_i,\ \text{etc.}$$

We then obtain the gravity-free rocket equation:

$$v_R(t) = v_R(t_i) + v_E\ \ell n \left[1 - \frac{R}{m_{Ri}}(t - t_i)\right]^{-1};\ t_i \le t \le t_f\,. \qquad (10)$$

This is plotted, for a particular set of rocket parameters, as a solid line in Fig. 2.

3. Modifications for Gravity

When a rocket rises vertically from the surface of the earth, against gravity, the additional force adds to the thrust and thus Eq. (4) becomes:

$$a_R = \frac{v_E R}{m_R} - g\,. \qquad (11)$$

Integrating and assuming $v_R(t_i) = 0,\ t_i = 0$, we obtain a revised Eq. (7):

$$v_R(t_f) = v_E\ \ell n \left(1 + \frac{m_F}{m_S}\right) - g t_f\,. \qquad (12)$$

Finally, Eq. (10) becomes:

$$v_R(t) = v_E\ \ell n \left(\frac{1}{1 - \dfrac{Rt}{m_{Ri}}}\right) - gt;\ 0 \le t \le t_f\,. \qquad (13)$$

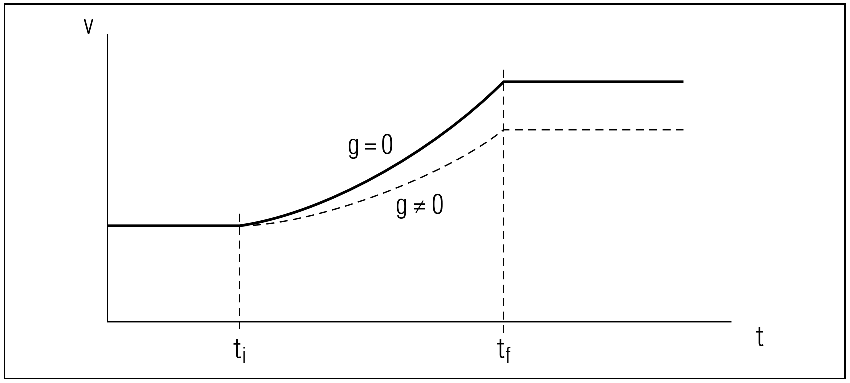


Figure 2. Rocket velocity as a function of time for gravity-free space (solid line) and for a rocket rising in a constant gravitational field (dashed line).

Note that if the burn rate is high, so that t_f is small, the gravity term in Eqs. (12) and (13) will be small. That is typically the case. Equation (13) is plotted as a dashed line in Figure 2.

Acknowledgments

M. Hunter and Steve Smith provided valuable feedback on an earlier version. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

Note: Problems 12-13 are also on this module's *Model Exam.*

1. A vertical rocket weighs 13 tons, of which 9.75 tons is fuel. Assume a constant fuel ejection velocity of 2 mi/s and assume that the effect of gravity is so small that it can be neglected. What is the speed of the rocket when the fuel has been exhausted?

2. A rocket whose weight is 3000 tons, when fueled up on the rocket pad, is fired vertically upward. At burnout 2780 tons remains. Gases are exhausted at a velocity of 165,000 ft/s relative to the rocket; both quantities are constant while the fuel is burning.

   a. What is the thrust?

   b. What is the speed of the rocket at burnout?

3. If the rocket in Problem 2 is fired in deep space (no external forces), what is its speed at burnout?

4. The 13 ton rocket in Problem 1 is remade as a 2-stage rocket. The first stage weighs 12 tons, of which 9 tons is fuel. The second stage weighs the remaining 1 ton, of which 0.75 ton is fuel. The second stage is fired after the fuel has been exhausted in the first stage and it has been decoupled. Again neglecting effects due to gravity, find the final velocity of the second stage. Note that you must apply the rocket equation twice, in succession. Compare to the final velocity of the one-stage rocket.

5. A boy with a pea shooter is standing on roller skates on a horizontal frictionless surface. The mass of the system (boy, skates, pea shooter, and peas) at a particular instant is M. At the same instant the boy is shooting peas of mass mp each whose velocity relative to the earth is v_p and the velocity of the peas with respect to the boy is v_r. All of these velocities are colinear. The number of peas per unit time is N. What is the average thrust on the boy due to the ejected peas at that instant?

6. Does a simple scaling-up of the fuel mass and rocket shell mass provide greater thrust? Greater velocity?

7. Given an exhaust velocity of $V_{ES} = 10,860$ mph and a fuel/fuel-container-plus-engine ratio of 90amount o fuel necessary to raise a payload of 1000 lb to the velocity necessary to escape earth's gravity ($V_{ES} = 25,000$ mph).

   Note: $r \equiv \frac{M_F}{M_{FC} + M_E} = 0.9$

   where M_{FC} is the mass of the fuel container and M_E is the mass of the engine. M_F is the mass of the fuel.

8. Show that each of the three quantities on the right hand side of Eq. (4) occurs in a reasonable position.

9. Show that the right hand side of Eq. (7) agrees with what one would expect as $M_F \to 0$ or $m_S \to \infty$.

10. Show that the right hand side of Eq. (6) agrees with what one would expect if $m_{RF} = m_{Ri}$.

11. Use the small-x approximations $(1 \pm x)^{-1} \approx (1 \mp x)$ and $\ell n(1 \pm x) \approx \pm x$ on Eq. (13) to show that the rocket's initial acceleration is:

    $a = \frac{v_E R}{m_{Ri}} - g.$

    Show that this agrees with what one would expect from Newton's second law and the thrust equation applied at time zero.

12. The earth escape velocity, 25,000 mph, is the upward velocity needed at the surface of the earth for eventual escape from the earth's gravitational pull. If the maximum fraction of a rocket's mass that can be devoted to fuel is 90%, determine the minimum exhaust velocity necessary for the rocket to reach escape velocity.

13. Show that the equation developed in Problem 1 above agrees with what one would expect if t is set equal to the initial time, t.

Brief Answers:

1. $2.77\,\text{mi/s} \approx 9,970\,\text{mph} = \text{mach } 13.4$
2. a. $F = 4.67 \times 10^8\,\text{lb} = 2.34 \times 10^5$ tons

   b. $v = 7,610\,\text{ft/s} = 5,190\,\text{mph} = \text{mach } 7.0$
3. $v = 12,600\,\text{ft/s} = 8,590\,\text{mph} = \text{mach } 11.6$
4. $v = v_E\ \ell n 13 = 5.13\,\text{mi/s} = 18,500\,\text{mph} = \text{mach } 25$; 85% greater.
5. $F = N v_r m_p$
6. Yes; no.
7. $m_F = m_p \left[(1 - e^{-v_{ES}/v_{EX}})^{-1} - r^{-1}\right]^{-1} = 14 \times 10^6$ lb.

12. 10,860 mph

13. $v_R(t_i) = v_R(t_i)$: check!

MODEL EXAM

1. See Output Skills K1-K3 on this module's *ID Sheet*.

2. The earth escape velocity, 25,000 mph, is the upward velocity needed at the surface of the earth for eventual escape from the earth's gravitational pull. If the maximum fraction of a rocket's mass that can be devoted to fuel is 90%, determine the minimum exhaust velocity necessary for the rocket to reach escape velocity.

3. Show that the equation developed in Problem l above agrees with what one would expect if t is set equal to the initial time, t.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, Problem 12.

3. See this module's *Problem Supplement*, Problem 13.