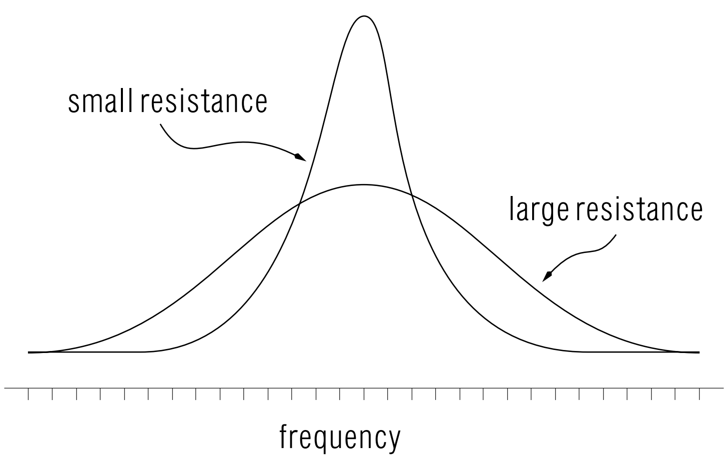
MISN-0-154

CIRCUIT RESONANCES



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

CIRCUIT RESONANCES
by
Peter Signell

Title: **Circuit Resonances**

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Input Skills:

1. Write down the equation governing voltage-driven 3-element LRC circuits (MISN-0-152).
2. Show that $q(t) = A\, e^{-Rt/(2L)} \sin(\omega_0 t + \alpha)$ with $\omega_0 \equiv 1/\sqrt{LC}$ and any A and α solves the undriven case: $0 = C^{-1}q + RI + L\dot{I}$ (MISN-0-29).

Output Skills (Knowledge):

K1. Show that $q(t) = q_t(t) + q_s(t)$ is a solution for the sinusoidally-driven series LRC circuit, where $q_t(t) = Ae^{-\gamma t}\sin(\omega_1 t + \alpha)$ and $q_s(t) = B(\omega)\sin[\omega t + \beta(\omega)]$ and the driving voltage is: $V_0\cos(\omega t)$. Show that $q_t(t)$ is a transient solution and q_s is a steady-state solution.

K2. Given a series LRC circuit driven by a sinusoidally-varying potential, $V_0\cos(\omega t)$, and given the steady-state solution, $q_s(t) = B(\omega)\sin[\omega t + \beta(\omega)]$, show or describe how one shows that: $\beta(\omega) = \tan^{-1}[(\omega_0^2 - \omega^2)L/(\omega R)]$ and $B(\omega) = (V_0\cos\beta)/(R\omega)$ or equivalent.

K3. Sketch phasor diagrams, and interpret them, to illustrate the phase relationships between voltages in the sinusoidally-driven series LRC circuit.

K4. Show or describe how one shows that the time-average steady-state power transferred into a circuit by a sinusoidally-varying potential is: $P_{ave}(\omega) = (V_0^2 R\omega^2/2)[L^2(\omega_0^2 - \omega^2)^2 + R^2\omega^2]^{-1}$.

K5. Sketch P_{ave} vs. ω in the vicinity of the resonant frequency of a sinusoidally-driven series LRC circuit both for a broad resonance and for a narrow one. Label each curve as to relative size of the resistance.

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New authors, reviewers and field testers are welcome.

CIRCUIT RESONANCES

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1. Introduction

The concept of “resonance” is one of the most important in all of science and technology. There are many mechanical processes where resonance is essential, other mechanical processes where it is to be avoided at all cost. Included in the former category are the parts of instruments which produce music; in the latter, the recording and playback apparatus. In the former, the vocal cords of a person speaking or a bird singing; in the latter, most parts of the ear of a person or a bird listening.

The same is true in electrical circuits: resonances can be useful for producing waves of a particular frequency as in television, radio, microwaves, radar, etc. It can also be unwanted, ocurring when some component fails, producing a huge current and perhaps a disastrous meltdown.

2. Transient and Steady-State

2a. Combining R, L, C, and the Driving Potential. We combine the potential drops across an inductor, a resistor, and a capacitor connected in series across a sinusoidally-varying driving voltage, shown pictorially in Fig. 1 and mathematically here:

$$V_0 \cos\omega t = C^{-1}q + RI + L dI/dt\,.$$

Indicating time derivatives by primes, we can rewrite this as:

$$Lq'' + Rq' + C^{-1}q = V_0 \cos\omega t\,, \tag{1}$$

This is a second order equation so the solution must satisfy Eq. (1) *and* must have two independent adjustable constants.

2b. Transient and Steady-State Parts. The solution to Eq. (1) must have two independent adjustable constants, yet we know intuitively that as time goes by the solution must settle down to a “steady-state” form dictated entirely by the driving potential [the right side in Eq. (1), V_{DR} in Fig. 1]. There can be a “transient” form at earlier times but its

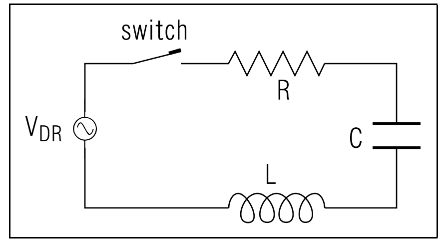

Figure 1. Switch is open for $t < 0$, switch is closed for $t > 0$. Here V_{DR} is an oscillating driving potential.

energy must gradually dissipate in the circuit resistance as time goes on, leaving only the steady-state form. Only the transient form can contain the two adjustable constants, since the long-term solution must depend only on V_0, ω, L, R, and C. We then write the solution for the charge on the capacitor as:

$$q(t) = q_t(t) + q_s(t)\,, \tag{2}$$

where, as it turns out, the “transient” part, q_t, is

$$q_t(t) = Ae^{-\gamma t}\sin(\omega_1 t + \alpha)\,, \tag{3}$$

and the “steady-state” part, q_s, is:

$$q_s(t) = B(\omega)\sin\left[\omega t + \beta(\omega)\right]\,, \tag{4}$$

where we have used these abbreviations: $\gamma \equiv R/(2L)$, $\omega_1 \equiv \sqrt{\omega_0^2 - \gamma^2}$, and $\omega_0 \equiv 1/\sqrt{LC}$.

2c. Examination of the Two Solutions. As one can see by inspection, the transient part, Eq. (3), dies away as time passes. The steady-state part, Eq. (3), just keeps on going without alteration. Any amount of the transient part can be present because A and α are set by one's values for charge and current at time zero and these are presumably under your control. You can even make the transient part zero by proper choice of the charge and current at time zero. As for the steady-state part, its amplitude B and phase β are independent of initial conditions and are simply functions of the driving voltage V_0 and the circuit parameters L, C, and R.

2d. How the Solutions Were Obtained. The solutions, Eqs. (3) and (4), were obtained by first separating Eq. (1) so as to show the “transient” and “steady-state” parts explicitly:

$$(Lq_t'' + Rq_t' + C^{-1}q_t) + \left(Lq_s'' + Rq_s' + C^{-1}q_s - V_0\cos\omega t\right) = 0\,. \tag{5}$$

The transient part of Eq. (5) is:

$$(Lq_t'' + Rq_t' + C^{-1}q_t) = 0\,. \tag{6}$$

and the steady-state part is:

$$Lq_s'' + Rq_s' + C^{-1}q_s - V_0\cos\omega t = 0\,. \tag{7}$$

These two quations are solved separately.

▷ Add the left sides of Eqs. (6) and (7) and see that they give the left side of (5), just as promised in Eq. (2). Do the same for the right sides.

It can be easily demonstrated that the transient equation, Eq. (6), has the solution shown in Eq. (3),[1] where A and α must be determined from $q(0)$ and $I(0)$.

3. Solving the Steady-State Equation

3a. Algebraic Method. We now substitute $q_s(t) = B\sin(\omega t + \beta)$ into Eq. (7) and get:

$$(C^{-1} - \omega^2 L)B\sin(\omega t + \beta) + R\,\omega B\cos(\omega t + \beta) - V_0\cos(\omega t) = 0. \tag{8}$$

Now we use the identity

$$\cos(A - B) = \cos A\cos B + \sin A\sin B\,,$$

[1]See Appendix A of this module.

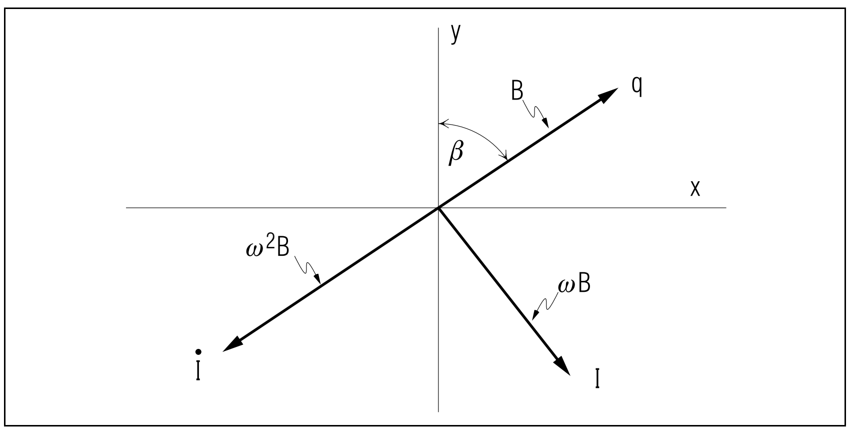

Figure 2. Charge and current phase relationships in Eq. (4) at $t = 0$.

which should be on instant recall, with $A = \omega t + \beta$ and $B = \beta$, to get:

$$\cos\omega t = (\sin\beta)\sin(\omega t + \beta) + (\cos\beta)\cos(\omega t + \beta)\,.$$

Put that into Eq. (8) and collect terms:

$$\left[\left(C^{-1} - \omega^2 L\right)B - V_0\sin\beta\right]\sin(\omega t + \beta) + (R\,\omega B - V_0\cos\beta)\cos(\omega t + \beta) = 0\,.$$

Now the sine and cosine are independent functions of time, so each of their constant coefficients must separately be zero in order to make the sum of the terms stay zero at all times. Then:

$$\begin{aligned}\left(C^{-1} - L\omega^2\right)B &= V_0\sin\beta\,,\\ R\omega B &= V_0\cos\beta\,.\end{aligned} \tag{9}$$

Dividing one of these two equations by the other gives $\tan\beta$, while summing the squares of the two equations gives B. Finally, we generally make the substitution

$$\omega_0 \equiv 1/\sqrt{LC}\,,$$

so the first of Eqs. (9) becomes:

$$\left(\omega_0^2 - \omega^2\right)LB = V_0\sin\beta\,,$$

3b. Phasor Method. First, since $q_s(t) = B\sin(\omega t + \beta)$, we get:

$$V_L = Lq_s'' = -L\omega^2 B\sin(\omega t + \beta)\,,$$

$$V_C = C^{-1}q_s = C^{-1}B\sin(\omega t + \beta)\,,$$

$$V_R = Rq_s' = R\omega B\cos(\omega t + \beta)\,,$$

so we can draw phasor diagrams at $t = 0$ for the charge/current relationships and these are as shown in Figs. 2 and 3.[2]

Finally, in Fig. 2 we draw a phasor diagram for the voltages in the circuit. Note that the vector sum of the voltage phasors for the resistor, the inductor, and the capacitor, must equal the vector phasor for the driving voltage. We take x- and y-components of these phasor equations and find that we have just the two equations of Eq. (9).

[2]See Appendix B of this module.

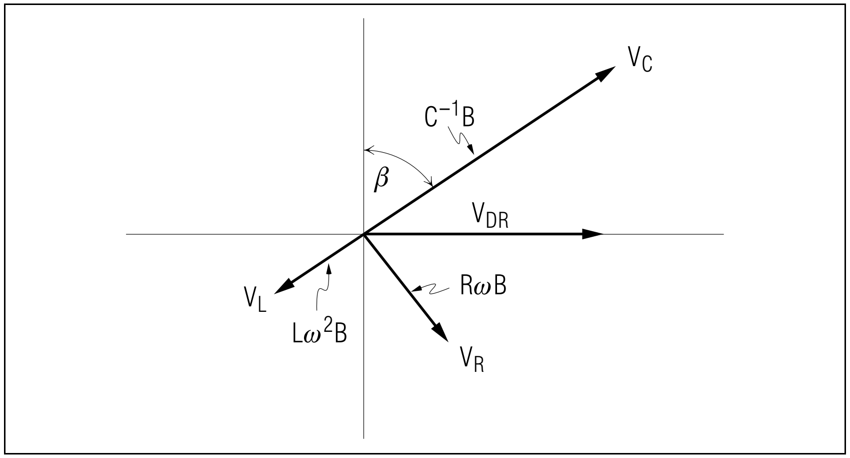

Figure 3. Voltage phase relationships in Eq. (4) at $t = 0$.

4. Average Power Dissipation

4a. Setting Up the Time-Average Integral. The time-average steady-state power transferred into the circuit from the driving voltage can be determined via the power-voltage-current relation:

$$P(t) = V(t)I(t)\,.$$

Then over one period $\mathcal{P}$ the average power is:

$$P_{ave} = \frac{1}{\mathcal{P}}\int_0^{\mathcal{P}} P(t)\,dt = \frac{1}{\mathcal{P}}\int_0^{\mathcal{P}} V_{DR}(t)I(t)\,dt\,,$$

where the driving voltage is used because we are trying to obtain the average power expended by that voltage source. Substituting $V_{DR}(t)$ and $I(t)$:

$$P_{ave} = \frac{1}{\mathcal{P}}\int_0^{\mathcal{P}} V_0(\cos\omega t)B\omega\cos(\omega t+\beta)\,dt\,.$$

Again use: $\cos(A+B) = \cos A\cos B - \sin A\sin B$, now with $A = \omega t$ and $B = \beta$:

$$P_{ave} = \frac{V_0\,\omega}{\mathcal{P}}\left[\cos\beta\int_0^{\mathcal{P}}\cos^2\omega t\,dt - \sin\beta\int_0^{\mathcal{P}}\cos\omega t\sin\omega t\,dt\right]. \qquad (10)$$

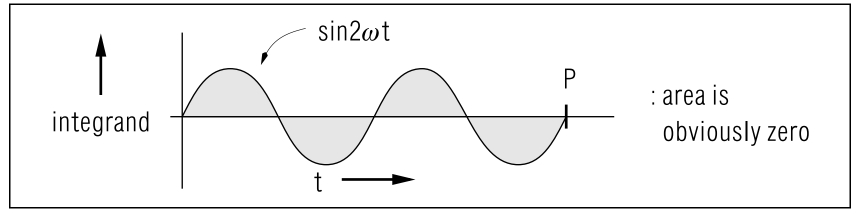

Figure 4. The integrand of the second integral in Eq. (10).

4b. Integrating. The second integral is zero, as can be easily seen by writing it as:

$$\int_0^{\mathcal{P}} \frac{1}{2}(\sin 2\omega t)\,dt\,,$$

and simply looking at the integrand (see Fig. 4). The value of the first integrand is $\mathcal{P}/2$ because the average value of $\cos^2$ is $1/2$.[3] Then:

$$P_{ave} = \frac{V_0 B\omega\cos\beta}{2}\,.$$

4c. Eliminating the Phase Angle. We can evaluate $\cos\beta$ from[4]

$$\cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+\dfrac{L^2(\omega_0^2-\omega^2)^2}{R^2\omega^2}}} = \frac{R\omega}{\sqrt{R^2\omega^2 + L^2(\omega_0^2-\omega^2)^2}}\,.$$

The final answer is then:

$$P_{ave} = \frac{V_0^2\,\omega^2 R}{2\left[L^2(\omega_0^2-\omega^2)^2 + R^2\omega^2\right]}\,.$$

5. Resonances

5a. Power Spectrums with Resonances. The "resonant" frequency is ω_0. If ω is swept through a range of frequencies, we get the power spectrum shown in Fig. 5.

[3]See Appendix C, this module, Trick 1.
[4]See Appendix C, Trick 2, for a slightly different approach.

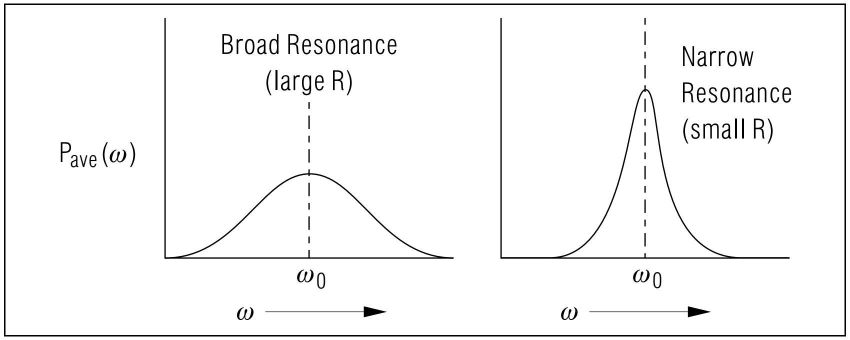

Figure 5. Power spectrums for broad and narrow resonances.

5b. Resonance Width (Approximate). It is easy to derive a formula for the width of the resonance in the "narrow width approximation." The width Γ of the resonance is defined at half the maximum resonance height h as shown in Fig. 6. Then:

$$P_{ave}(\omega_{1/2}) = \frac{1}{2} P_{ave}(\omega_0)\,.$$

Substituting in both sides:

$$\frac{V_0^2\omega_{1/2}^2 R}{L^2(\omega_0^2-\omega_{1/2}^2)^2 + R^2\omega_{1/2}^2} = \frac{1}{2}\cdot\frac{V_0^2\omega_0^2 R}{R^2\omega_0^2}\,,$$

which results in:

$$\omega_0^2 - \omega_{1/2}^2 = \pm R\,\omega_{1/2}/L\,.$$

Factor the left hand side:

$$\omega_0^2 - \omega_{1/2}^2 = (\omega_0 - \omega_{1/2})\cdot(\omega_0 + \omega_{1/2})\,,$$

and use the narrow width approximation

$$\omega_0 \approx \omega_{1/2}\,,$$

to get:

$$\omega_0 + \omega_{1/2} \approx 2\omega_0\,,$$

$$\omega_0 - \omega_{1/2} = \pm 2R/2L\,,$$

$$\omega_{1/2} = \omega_0 \pm R/2L\,.$$

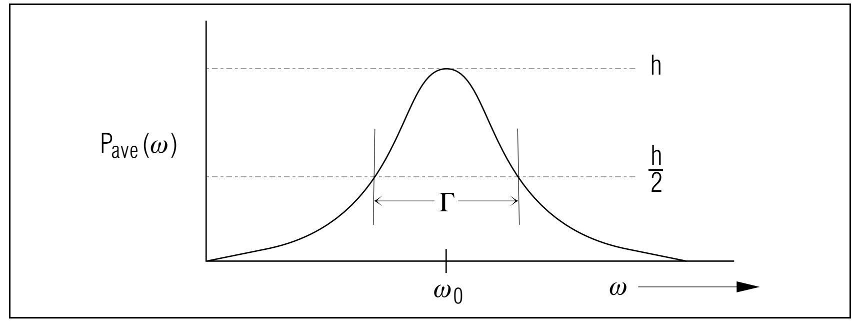

Figure 6. Power spectrum showing the definition of a resonance's width Γ.

The full width at half maximum is then:

$$\Gamma = R/L\,.$$

This shows that the width is directly proportional to the circuit resistance. The average power put into the circuit by the driving voltage source is often written:

$$P_{ave}(\omega) = \frac{V_0^2\Gamma/(8L)}{(\omega-\omega_0)^2 + (\Gamma/2)^2}\,,$$

but you should realize that this is only valid in the narrow-width approximation. An interesting relation between the height and width of the resonance can be obtained by evaluating the height at ω_0:

$$P_{ave}(\omega_0) = \frac{V_0^2}{2\Gamma L}\,,$$

so the height varies inversely as the width.

5c. Resonances as Complex-Plane Poles. A broad resonance is the shoulder of a far-away pole in the complex plane. As the circuit resistance is decreased, the pole moves closer to the real axis. This causes the shoulder to narrow and heighten. In fact, the width of the resonance is the distance of the pole from the real axis. These matters are discussed and illustrated elsewhere.[5]

[5] See "Resonances and Poles: Relationship Between the Real and Imaginary Worlds" (MISN-0-49).

Acknowledgments

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A. The Transient Solution

(for those interested)

1. Let $q = Ae^{-\gamma t}\cos(\omega t + \alpha)$, where γ is some unknown function of L, R, and C.

2. Differentiate q to get:

   $$I = -\gamma Ae^{-\gamma t}\cos(\omega t + \alpha) - \omega Ae^{-\gamma t}\sin(\omega t + \alpha)\,.$$

3. Differentiate I to get:

   $$\dot{I} = -\gamma I + \omega A\gamma e^{-\gamma t}\sin(\omega t + \alpha) - \omega^2 Ae^{-\gamma t}\cos(\omega t + \alpha)\,.$$

4. Substitute these into the master equation and the equality reduces to: $\gamma = R/(2L)$, $\omega^2 = \omega_0^2 - \gamma^2$, $\omega_0^2 = 1/(LC)$.

B. Constructing Phasor Diagrams

(a review)

1. A phasor diagram such as the one in Fig. 2 is a way of seeing the phase relationships between circuit quantities.

2. Each circuit voltage or current is represented by a vector that rotates *counterclockwise* at angular velocity ω. One can think of all the vectors on a single diagram as being locked together as they rotate.

3. The physical value of any of the represented quantities is the projection of that quantity's phase vector on the real axis. A projection on the y-axis has no physical meaning. Thus the diagram is not to be confused with a real vector diagram: the "vectors" on it are imaginary constructs.

4. The length of a phasor is the *maximum* value the quantity will have when watched over a complete cycle.

5. The angle between a phasor and the positive x-axis is the phase angle of that quantity at that moment.

6. Notice that, in Fig. 1, the q is 90° ahead of the I, and I is 90° ahead of $\dot{I}$. This means, for example, that the voltage on the capacitor peaks a quarter cycle ahead of when the voltage on the resistor peaks, and that is a quarter cycle ahead of when the voltage on the inductor peaks.

C. Some Tricks

(for those interested)

Trick 1. Here is an easy way to see that the value of $\sin^2$ or $\cos^2$ over any number of complete half cycles is one-half. We make use of the fact that the average values of $\sin^2$ and $\cos^2$, over any number of complete half cycles, are obviously equal (see Fig. 7). Then:

$$\int_0^{2\pi} \cos^2 x\,dx = \int_0^{2\pi} sin^2 x\,dx\,,$$

which can be used to evaluate the average value of $\cos^2 x$:

$$\begin{aligned}(\cos^2 x)_{ave} &= \frac{\int_0^{2\pi}\cos^2 x\,dx}{2\pi} = \frac{1}{2}\cdot\frac{\int_0^{2\pi}\cos^2 x\,dx}{2\pi} + \frac{1}{2}\cdot\frac{\int_0^{2\pi}\sin^2 x\,dx}{2\pi}\\ &= \frac{\int_0^{2\pi}(\cos^2 x + \sin^2 x)\,dx}{4\pi} = \frac{\int_0^{2\pi}(1)\,dx}{4\pi} = \frac{2\pi}{4\pi} = \frac{1}{2}\,.\end{aligned}$$

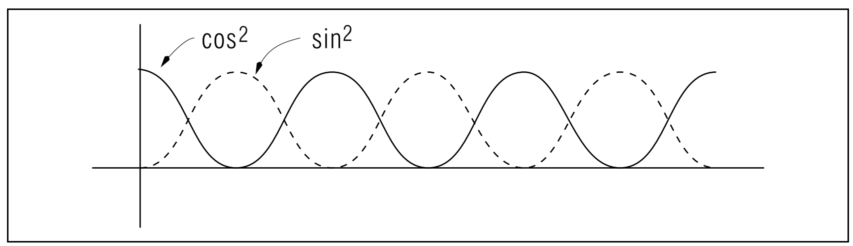

Figure 7. Comparison of $\sin^2$ and $\cos^2$ for use in evaluating the first integral in Eq. (10).

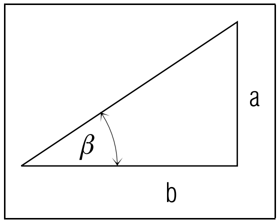

Figure 8. A triangle for evaluating complex trigonometric relations.

Trick 2. Rather than working out $\cos\beta$ from $\tan\beta$ as in Sect. 4c, physicists often use a triangle. If $\tan\beta = a/b$ than a and b can be drawn as the legs of a right-angle triangle shown in Fig. 8. The hypotenuse is obviously $\sqrt{a^2+b^2}$ and so $\cos\beta = b/\sqrt{a^2+b^2}$. In our case:

$$\tan\beta = \frac{L(\omega_0^2 - \omega^2)}{R\omega},$$

hence

$$\cos\beta = (R\omega)/\sqrt{R^2\omega^2 + L^2(\omega_0^2-\omega^2)^2}\,.$$

MODEL EXAM

1. See Output Skills K1-K5 in this module's *ID Sheet*. The actual exam may contain any selection of these skills or all of them.

Brief Answers:

1. See this module's *text*.