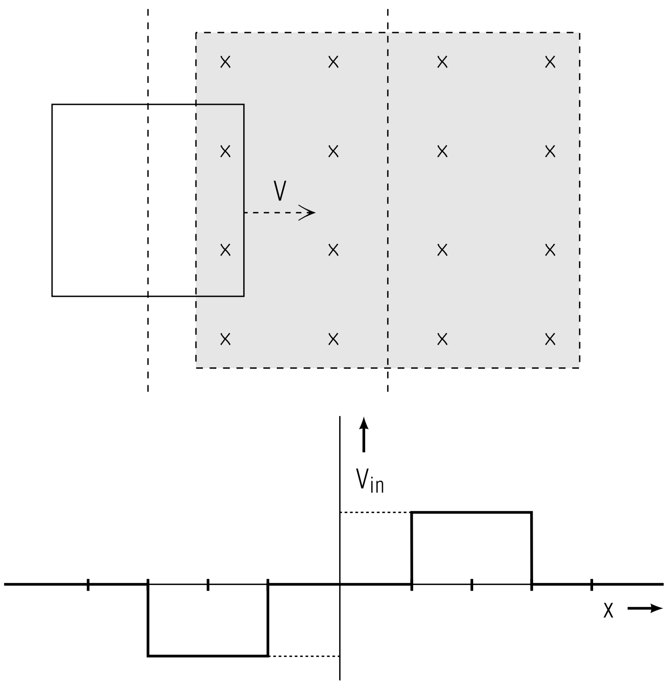
MISN-0-142

MAGNETIC INDUCTION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

MAGNETIC INDUCTION
by
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Title: **Magnetic Induction**

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Input Skills:

1. Vocabulary: energy-power-time relation (MISN-0-20), electric field (MISN-0-115), battery(MISN-0-117), power (electrical) (MISN-0-118), voltage, resistor (MISN-0-119), magnetic field, Lorentz force (MISN-0-122), surface integral (MISN-0-132), line integral, Ampere's law (MISN-0-138).

Output Skills (Knowledge):

K1. Vocabulary: Faraday-Henry law, flux, induced current, induced potential difference, induced voltage, Lenz's law.

K2. Write down the Faraday-Henry Law, relating the line-integral of the induced electric field to the time rate of change of the surface integral of the magnetic field across a surface bounded by the path of the line-integral, taking care to explain the sign convention. Explain in words what the physical phenomenon is that is described by this equation.

Output Skills (Problem Solving):

S1. Determine the voltage induced around given closed paths when a changing magnetic field in the region is specified.

S2. Determine the voltage induced around given closed paths when the velocity of the path through a uniform static magnetic field is specified.

S3. Use Lenz's law to determine the directions of induced voltages, currents, and magnetic fields.

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New authors, reviewers and field testers are welcome.

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1. Introduction

1a. Overview. When magnetic fields and/or electric fields change with time, effects occur which are profoundly different from what would be expected from just the static properties of these fields. When electric fields change with time, magnetic fields are produced just as would be produced by currents. When magnetic fields change with time, electric fields are produced just as would be produced by charges. The fields so-produced are called "induced" fields and they simply add to the fields produced by charges and currents. In this module we study: (1) moving circuits in stationary magnetic fields, so the Lorentz Force provides the induction; and (2) stationary circuits in time-changing magnetic fields so the induction is governed by the so-called "Faraday-Henry" law.

1b. Magnetic Flux. Throughout the study of magnetic induction we make use of the concept of "the magnetic flux through a surface S bounded by a directed line L," written Φ_{SL} and defined by:

$$\Phi_{SL} \equiv \oint_{SL} \vec{B} \cdot d\vec{S} \,. \tag{1}$$

The relationships between these quantities are shown in Fig. 1. The word "flux," strictly speaking, merely refers to value of the integral on the right side of Eq. (1). However, the word "flux" originated with people who liked to think of it as a sort of measure of the number of (ficticious) magnetic field lines that go through the surface of integration.

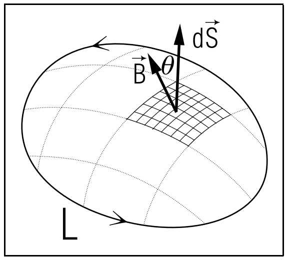


Figure 1. The angle θ is defined as the angle between the magnetic field $\vec{B}$ and the vector $d\vec{S}$ whose direction is normal to the surface element of area dS.

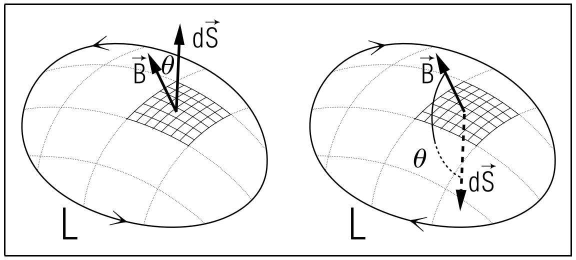


Figure 2. The case where $d\vec{S}$ is determined by L: (a) for one choice of the direction L; and (b) for the other choice.

1c. The Flux Integrand. To construct the integrand in Eq. (1), one needs to take the scalar product of the two vectors $\vec{B}$ and $d\vec{S}$ and for that their directions and magnitudes must be known. The vector $\vec{B}$ is normally known in any particular application but we *must* determine the direction of the infinitesimal vector surface element $d\vec{S}$ by examining that element and its relationship to the directed line L. After all, there are two sides to the surface and each side has a unit vector normal to the surface at that point. Note that the unit vector $\hat{n}$, normal to the surface, enters Eq. (1) in the form: $d\vec{S} = \hat{n}\, dS$. The two normal unit vectors on opposite sides of the surface point in exactly opposite directions. The scalar product in Eq.(1) will be positive with one choice for positive $d\vec{S}$, negative with the other choice. However, the sign of the integrand *must* be determined correctly because the sign has important physical consequences.

1d. The Direction of Positive Flux. The positive flux direction for a surface S is tied to the positive direction around the line L that borders the surface. Once you choose one of those two directions, the other is thereby fixed. You may choose one of them in any manner that appeals to you, but then you must define the other in conformity with the choice you just made. The directions of L and S are related the same way that the magnetic field surrounding a wire is related to the direction of the current in the wire, or in the same way that the direction of the current in a loop of wire is related to the magnetic field at the center of the loop, as ilustrated in Fig. 2.

▷ There is a loop of area $4.0\,\mathrm{m}^2$ in the x-z plane: take the surface the loop encloses as a flat disc in the plane. The magnetic field is, at some time, say, $\vec{B} = 12\,\mathrm{T}\,\hat{y}$ at all space points enclosed by the loop. Show that the magnetic flux through the loop is just the area times the field: $\Phi_{SL} = 48\,\mathrm{T\,m}^2$.

▷ If the field was tilted so it was at an angle θ to the normal to the surface, then Eq. (1) shows that the flux through the surface would be: $\Phi_{SL} =$

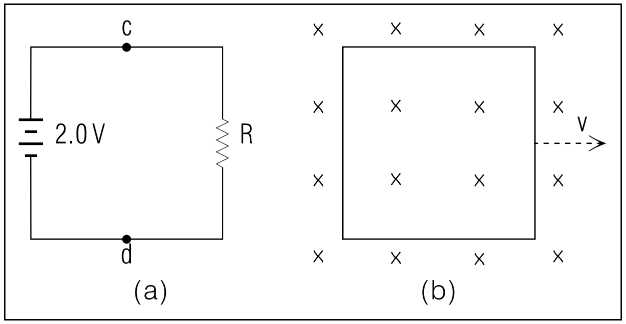


Figure 3. The voltage drop around a complete path: (a) zero for a non-induced voltage; (b) non-zero for an induced voltage.

$48 \cos\theta$ T m^2. Flux is sometimes given the unit "weber," abbreviated W: W $\equiv$ T m^2.

1e. Induced Voltage. When magnetic effects induce an electric field in a circuit, that electric field can be integrated along the circuit to obtain the induced potential difference:

$$V_{ind} = -\int \vec{E}_{ind} \cdot d\vec{\ell}\,. \tag{2}$$

This induced potential difference is called the "induced voltage" for short, and it will produce a current in the circuit if the circuit resistance is less than infinite. The induced voltage represents the work per unit charge that the electric field is willing to do. It is non-zero around a complete closed path, which is quite a difference from ordinary voltages (such as those supplied by batteries) that *are* zero around a closed path. We illustrate this in Fig. 3 with a specific battery-driven circuit and a specific induction-driven circuit where a magnetic field is changing with time. In Fig. 3a the voltage drop clockwise from c to d is 2.0 V and the voltage drop clockwise from d to c is -2.0 V (a voltage *rise*). Thus the voltage drop around the total circuit, clockwise from c to d to c, is zero. In Fig. 3b the voltage drop from any one point clockwise around the circuit and back to the original point is *not* zero and is, in fact, the *induced voltage*.

1f. Induced Voltage and Flux Change Rate. When the magnetic flux, through the surface that is surrounded by a loop, changes with time, the induced voltage equals the rate of change of the magnetic flux through any surface bounded by the loop:

$$V_{ind} = \frac{d\Phi}{dt}\,. \tag{3}$$

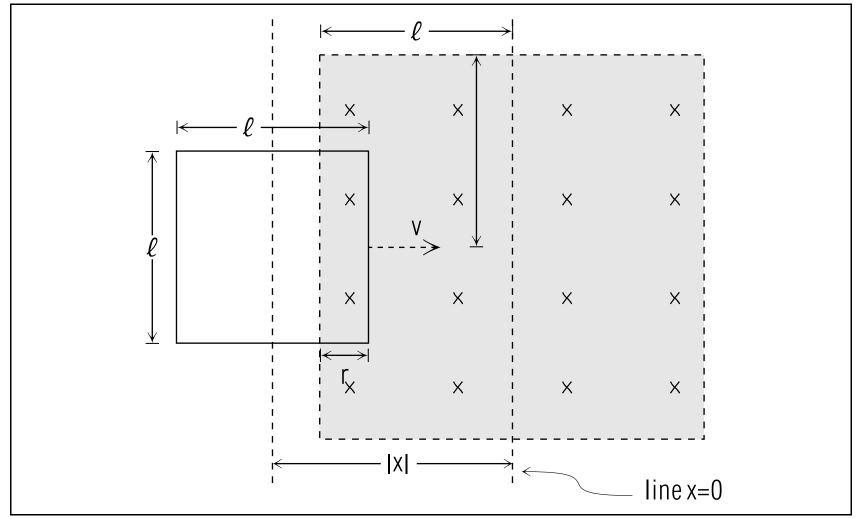


Figure 4. A square-loop circuit enters a region of constant magnetic field that is directed into the page.

1g. The Power Transferred. If the magnetic field is constant but the circuit is moving (the Lorentz-Force case) the power dissipated in the circuit resistance comes from the mechanical energy needed to move the circuit in the field. If the circuit is stationary but the magnetic field is changing with time (the Faraday-Henry case) the power dissipated in the circuit resistance comes from the energy that must be expended to change that field.

2. Example of Lorentz-Force Induction

2a. A Loop Enters A Magnetic Field. In Fig. 4 we show an example of Lorentz force induction. There a loop circuit traveling at speed v enters a region of constant magnetic field B. Note that x is the distance from the center line of the loop to the center line of the region of magnetic field; and that the length of each side of the loop is ℓ and the width of the magnetic field is 2ℓ. Until the leading edge of the loop reaches the field there is no Lorentz force on the mobile charges in the wire that constitutes the loop. When $x = -3\ell/2$ the wire enters the field and there is a Lorentz force on the mobile charges in part of the loop that has entered the field. That force is: $F = q\,v\,B$ so the induced electric field, the force per unit charge, is $E_{ind} = v\,B$. Integrating this along the leading edge of the loop,

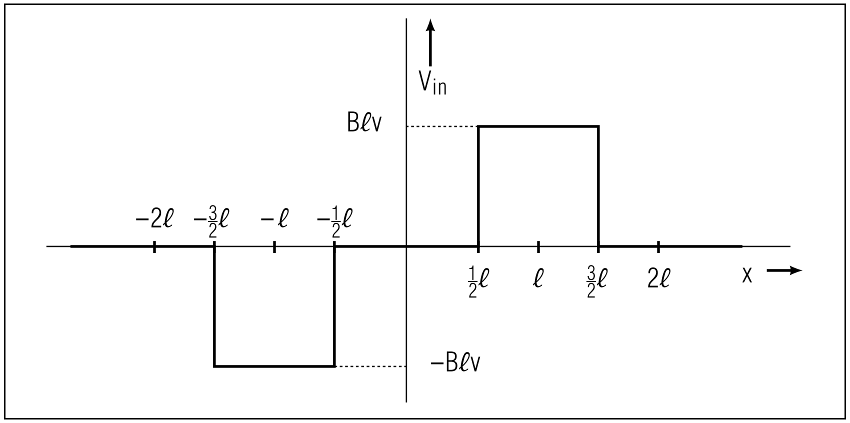


Figure 5. The induced potential difference around the loop of Fig. 4 as a function of the position of the loop.

we get for the induced potential of point b with respect to point a:

$$V_{ind,ba} = -\ell\, v\, B\,. \tag{4}$$

Thus any induced current will flow from a to b (from higher potential to lower, in the direction of the Lorentz Force).

2b. The Loop Travels Through the Magnetic Field. As the loop travels across the magnetic field the induced potential difference along the top and bottom arms of the loop will be zero as is readily seen from the vector form of the Lorentz force. The potential difference between the ends of the trailing edge of the loop will be the same as for the leading edge:

$$V_{ind,cd} = -\ell\, v\, B\,. \tag{5}$$

As we make a complete circuit around the loop the induced potential differences along the two ends oppose each other and the net potential difference is zero:

$$V_{ind,loop} = 0\,. \tag{6}$$

2c. The Loop Leaves the Magnetic Field. When the leading edge of the coil reaches the far edge of the field and emerges into a region of zero magnetic field, its induced potential drops to zero and the loop potential difference is:

$$V_{ind,dc} = +\ell\, v\, B\,, \tag{7}$$

which would cause current to flow around the loop in a clockwise direction. Fig. 5 shows a plot of the loop potential difference, V_{dcba}, vs. x.

2d. The Time Rate of Change of the Flux. The induced potential difference is simply related to the time rate of change of the magnetic flux enclosed by the loop:

$$V_{ind,dcba} = \dot{\Phi}\,. \tag{8}$$

This comes about because the magnetic field is constant so the flux integral, Eq. (1), is just $\Phi = B\, dA/dt$ where A is the area of the loop containing non-zero B. That area is $A = \ell r$ so $dA/dt = \ell dr/dt = \ell\, v$. Then $\dot{\Phi} = B\,\ell\, v$ and Eq. (8) is proved. We will find that Eq. (8) is not restricted to Lorentz force cases but is also true where the loop is stationary (not moving) in the field but the magnetic field itself is increasing or decreasing with time.

3. Induction by Time-Changing B

3a. The Faraday-Henry Law. Time-changing magnetic fields produce induced electric fields, hence potential differences, in a manner described by the so-called Faraday-Henry law. This law relates the line integral of the induced electric field to the time-derivative of the surface-integrated magnetic field:

$$\oint_{LS} \vec{E}_{ind} \cdot d\vec{\ell} = -\frac{d}{dt} \oint_{SL} \vec{B} \cdot d\vec{S}\,. \tag{9}$$

Now we can substitute $(-V_{ind})$ for the left side and $(-\dot{\Phi})$ for the right side to get the simpler-appearing form:

$$V_{ind,LS} = \dot{\Phi}_{SL}\,. \tag{10}$$

Note that this equation is exactly the same as the equation for Lorentz-force induction, Eq. (8).

3b. Example: Linearly Increasing B. Suppose a coil having N turns and area A is placed perpendicular to a magnetic field that is increasing linearly with time:

$$B = B_0 + B_1\, t\,,$$

where B_0 and B_1 are constants. The induced voltage is, by Eq. (10):

$$V_{ind} = \dot{\Phi} = N\, A\, B_1\,.$$

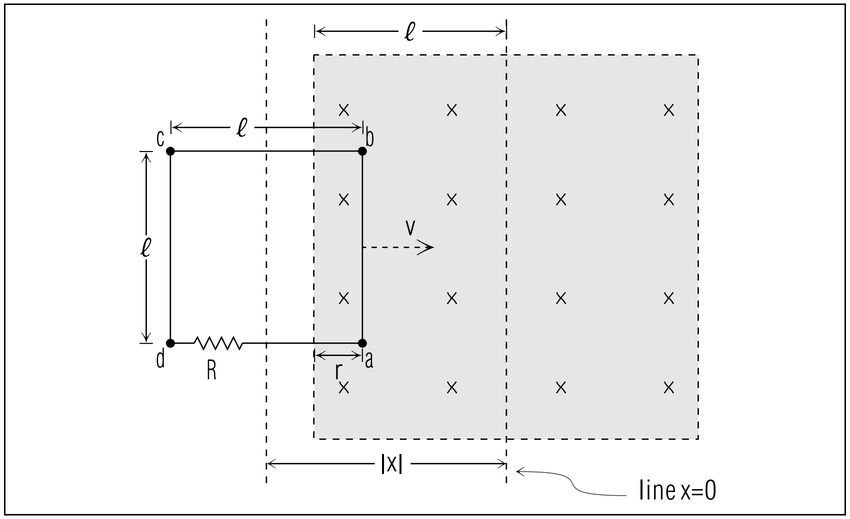


Figure 6. As Fig. 4 but with circuit resistance shown explicitly.

4. Lenz's Law

Lenz's law states that magnetic induction is always in such a direction as to tend to oppose the change that produced the induction. For example, as the loop in Fig. 4 enters the magnetic field the induced potential difference is such as to produce a *counter-clockwise* current in the loop. That counterclockwise current will itself produce a magnetic field that comes out of the page and so to some extent opposes the into-the-page continuing increase in the original loop flux. Similarly, the *clockwise* current induced as the loop leaves the field itself produces a magnetic field that is into the page: this tends to oppose the continuing decrease that is taking place in the loop flux.

Lentz's law also states that if a time-changing potential difference causes a time-changing magnetic field, then the time-changing magnetic field produces an electric field and a resulting current flow that will oppose the change in the original potential difference.

Similarly, a changing current induces a current that tends to oppose the continuing change in the original current.

5. Self-Inductance, Inductors

The term "inductor" refers to a circuit element that usually has the shape of a solenoid, a torus, or simply many loops of wire glued together. A current flowing through an inductor of course sets up a magnetic field so changes in the current result in changes in the magnetic field. Those in turn produce an induced voltage drop in the element, a voltage drop that opposes the change in the current. The magnitude of the induced voltage is proportional to the time-rate-of-change of the current, as we have seen, so we write:

$$V_{ind} = L\,\frac{dI}{dt}\,, \tag{11}$$

where L is determined solely from the geometry of the inductor and is called the inductor's *inductance*. The S.I. unit of inductance is the "henry," abbreviated H: $\mathrm{H} \equiv \mathrm{V\,s\,A^{-1}}$ as can be seen from Eq. (11).

Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **Faraday-Henry law**: the integral form of a Maxwell equation, stating that the line integral of the electric field around a loop is equal to the negative of the time derivative of the integral of the normal component of the magnetic field (integrated over the surface bounded by the loop): $\oint_{LS} \vec{E}\cdot d\vec{\ell} = -(d/dt)\int_{LS}\vec{B}\cdot d\vec{S}$. This law is often quoted in its short form: the voltage induced in the loop equals the time derivative of the magnetic flux through the surface bounded by the loop: $V_{ind} = \dot{\Phi}$.

- **flux**: denoted Φ; the integral, over a surface with a particular boundary, of the component of the field normal to the surface. For magnetic flux: $\Phi_{SL} = \int_{SL}\vec{B}\cdot d\vec{S}$.

- **induced current**: the current produced in a circuit as a result of an induced potential difference (which see). The usual relations between voltage, resistance, and current apply.

- **induced potential difference**: the total distributed potential difference encountered in traversing a circuit due to motion of the circuit in a magnetic field or due to the circuit being in a time-changing magnetic field.
- **induced voltage**: just another name for "induced potential difference" (which see).
- **self-inductance**: the negative of the induced voltage around a loop divided by the time-rate-of-change of magnetic flux through any surface bounded by that loop: $L = -V_{induced}/\dot{\Phi}$. The minus sign shows that the induced voltage opposes the change in the flux. The SI unit of inductance is the henry, abbreviated H: $\mathrm{H} \equiv \mathrm{V\,s\,A^{-1}}$.
- **inductor**: a circuit element whose purpose is to provide self-inductance, the electrical circuit analog of mechanical inertia (mass). An inductor is usually in the shape of a solenoid or a toroid. The self-inductance of an inductor depends on the geometry of the inductor, and the magnetic susceptibility of the materials of which the inductor is constructed. Inductors in electronic circuits typically are in the mH range.
- **Lenz's law**: a law stating that induced phenomena oppose the inducing phenomena. For example, an induced current produces a magnetic field which opposes the changes in the magnetic field that caused the current to be induced.

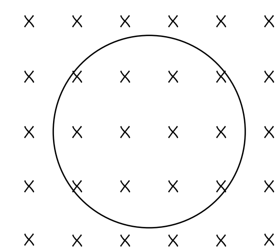
PROBLEM SUPPLEMENT

Note: Problems 1-3 also appear in this module's *Model Exam.*

1. A circular loop, of radius $R = 0.20$ meters, in the plane of the page is in a region where a magnetic field $\vec{B}$ is directed into the page (see sketch). The magnitude of $\vec{B}$ varies with time according to:

$$|\vec{B}| = B_0 - At^2,$$

where $B_0 = 2.0\,\mathrm{T}$ (teslas) and $A = 0.30\,\mathrm{T/s^2}$ are constants.



$\vec{B}$ directed into page

a. Find the magnitude and direction of the induced potential difference, V_{ind}, around this loop at the instant $t = 0.20$ sec. [F]

Is V_{ind} constant with time? [A]

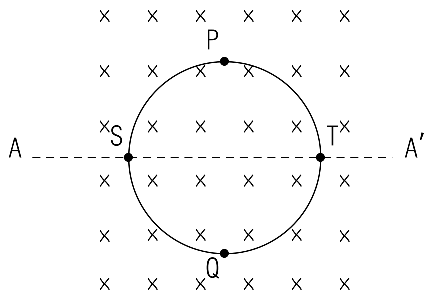
Check that the dimensions of V_{ind} follow correctly from the dimensions of what goes into your calculations. [H]

b. Find the magnitude and direction of the tangential component of the electric field vector at any point on this loop. [J]

c. What can the Faraday-Henry Law tell you about the normal (radial) component of $\vec{E}$ at any point on the loop? [E]

What does symmetry tell you? [I]

2.



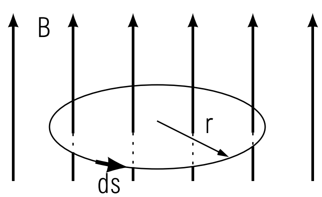
This same loop is placed in a region of uniform field $\vec{B}$, $|\vec{B}| = 2.0\,\mathrm{T}$, directed into the page. The loop is rotated about the axis AA' so that point P comes out of the page, point Q goes into the page. In 0.02 sec the loop is flipped over so that P is where Q was and vice versa.

a. What is the average V_{ind} induced around this loop? [B]

b. What is the direction of this V_{ind}? (Would it drive the induced current in the direction SPT or TPS? [D]

3. As shown in the figure, lines of magnetic field are directed perpendicular to a circular loop of wire of radius r.

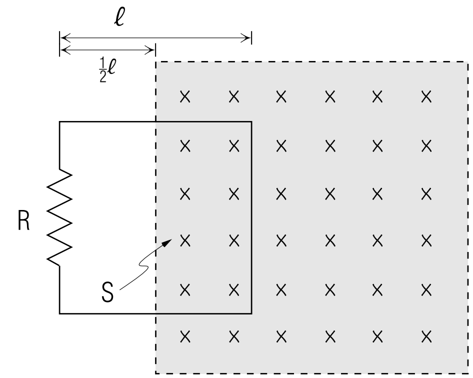
a. If the magnitude of the field is given by $B = B_0 e^{-t/\tau}$, where B_0 and τ are constants, what is the V_{ind} induced in the loop as a function of time? [G] (In the figure, $d\vec{s}$ denotes the positive direction around the loop.)



b. Answer the same question if the loop is titled through 60.0°, [L] or 90.0°. [K]

4. Using the definitions of the tesla (p. 359) and the volt, in terms of fundamental dimensions, show that both sides of the Faraday-Henry Law have the same units. [C]

5.



A square loop of wire is stationary within a time-dependent magnetic field that is confined to a square region of space. The length of one side of the loop is ℓ and the loop extends $\ell/2$ into the field (see sketch). The magnetic field varies with time as $B = B_0 sin(\omega t)$. Use these values to answer the questions below: $B_0 = 1.0\,\mathrm{T}$, $\omega = 2.0\pi/\mathrm{s}$, $R = 1.0\,\Omega$ and $\ell = 1.0\,\mathrm{m}$.

a. At $t = 3.5\,\mathrm{s}$ what is V_{ind}? [M]

b. At that same time what is the power dissipation through the resistor? [O] *Help: [S-3]*

c. What is the work done by the field from $t = 0.0\,\mathrm{s}$ to $t = 3.5\,\mathrm{s}$? [N] *Help: [S-2]*

Brief Answers:

A. No.

B. 25 volts. *Help: [S-1]*

C. See text.

D. Again, use Lenz's Law to find that the direction of decreasing V_{ind}, the "downhill" direction for the current, the direction the current will move, is SPT.

E. Nothing, in the case of this circular path.

F. 0.015 volts, clockwise.

G. $\dfrac{\pi r^2 B}{\tau}$.

Lenz's Law verifies, direction same as $d\vec{s}$.

Dimensions: $\mathrm{m^2\,T\,s^{-1}} = \mathrm{kg\,m^2\,s^{-2}\,C^{-1}} = \mathrm{N\,C^{-1}} =$ volt

H. See definition of the dimension of tesla, p. 359 of AF.

I. Symmetry tells you that whatever E_n (the component of $\vec{E}$ normal to page) is, it is the same at every point on the circle. Because the change in B is entirely perpendicular to the page, E_n in fact is zero. (Refer to Lenz's Law in this module's *text*).

J. $E_t = 0.012\,\mathrm{volt/m} = 0.012\,\mathrm{N/C}$.

K. Zero.

L. $\dfrac{\pi r^2 B}{2\tau}$.

M. $V_{ind} = -\dfrac{\ell^2 \omega B_0}{2}\cos(\omega t) = \pi\,\mathrm{V}$

N. $W = \dfrac{1}{R}\cdot\left(\dfrac{\ell^2}{2}\omega B_0\right)^2\left[\dfrac{t}{2} + \dfrac{1}{4\omega}\sin(2\omega t)\right] = 1.75\pi^2\,\mathrm{J}$ *Help: [S-3]*

O. $P = V_{ind}^2/R = \dfrac{\pi^2\,\mathrm{V}^2}{1.0\,\Omega} = \pi^2\,\mathrm{A\,V} = \pi^2\,\mathrm{W}$ *Help: [S-2]*

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS, Problem 2a)*

The average value of a sine function over a half cycle is:

$$\overline{\sin x} = \frac{\int_0^\pi \sin x\,dx}{\int_0^\pi dx} = \frac{|-\cos x|_0^\pi}{\pi} = \ldots$$

S-2 *(from PS, Problem 5b)*

The power-voltage-resistance relation is referenced in this module's *ID Sheet*.

S-3 *(from PS, Problem 5c)*

The energy-power-time relation is referenced in this module's *ID Sheet*.

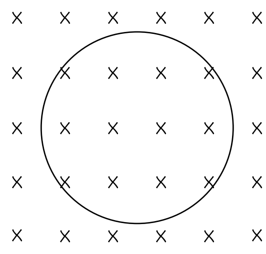
MODEL EXAM

1. See Output Skills K1-K2 in this module's *ID Sheet*. The actual exam may have one or more of these skills, or none.

2. A circular loop, of radius $R = 0.20$ meters, in the plane of the page is in a region where a magnetic field $\vec{B}$ is directed into the page (see sketch). The magnitude of $\vec{B}$ varies with time according to:

$$|\vec{B}| = B_0 - At^2,$$

where $B_0 = 2.0\,\mathrm{T}$ (teslas) and $A = 0.30\,\mathrm{T/s^2}$ are constants.



a. Find the magnitude and direction of the induced potential difference, V_{ind}, around this loop at the instant $t = 0.20$ sec.

Is V_{ind} constant with time?

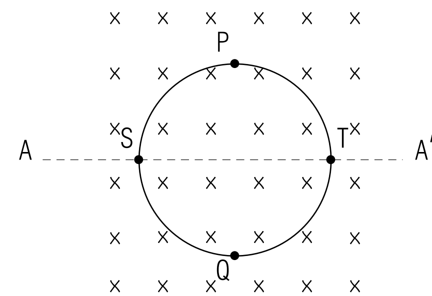
Check that the dimensions of V_{ind} follow correctly from the dimensions of what goes into your calculations.

b. Find the magnitude and direction of the tangential component of the electric field vector at any point on this loop.

c. What can the Faraday-Henry Law tell you about the normal (radial) component of $\vec{E}$ at any point on the loop?

What does symmetry tell you?

3.



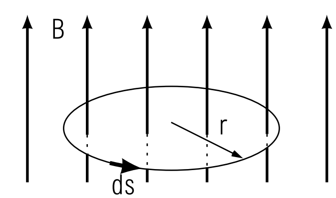
This same loop is placed in a region of uniform field $\vec{B}$, $|\vec{B}| = 2.0\,\mathrm{T}$, directed into the page. The loop is rotated about the axis AA' so that point P comes out of the page, point Q goes into the page. In 0.02 sec the loop is flipped over so that P is where Q was and vice versa.

a. What is the average V_{ind} induced around this loop?

b. What is the direction of this V_{ind}? (Would it drive the induced current in the direction SPT or TPS?

4. As shown in the figure, lines of magnetic field are directed perpendicular to a circular loop of wire of radius r.

a. If the magnitude of the field is given by $B = B_0 e^{-t/\tau}$, where B_0 and τ are constants, what is the V_{ind} induced in the loop as a function of time? (In the figure, $d\vec{s}$ denotes the positive direction around the loop.)



b. Answer the same question if the loop is titled through 60.0°, or 90.0°.

Brief Answers:

1. See this modules's *text*.

2. See this module's *Problem Supplement*, problem 1.

3. See this module's *Problem Supplement*, problem 2.

4. See this module's *Problem Supplement*, problem 3.