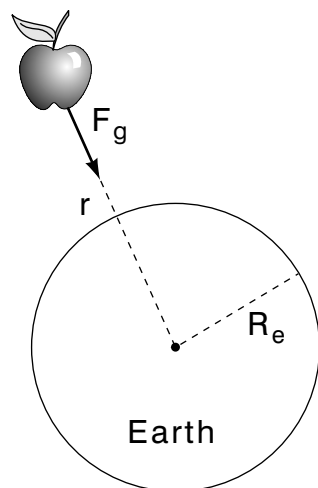


NEWTON'S LAW OF UNIVERSAL GRAVITATION



NEWTON'S LAW OF UNIVERSAL GRAVITATION

by
Paul M. Parker

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Title: **Newton's Law of Universal Gravitation**

Author: Paul M. Parker, Dept. of Physics, Mich. State Univ

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Input Skills:

1. Vocabulary: mass, force (MISN-0-62); center-of-mass (MISN-0-16).
2. In two dimensions, add a series of vectors graphically and analytically (MISN-0-2).
3. Given a vector's components in two dimensions, determine the magnitude of the vector and the angle it makes with each of the two axes of a cartesian coordinate system, and conversely (MISN-0-2).
4. Calculate the centripetal force acting on an object undergoing uniform circular motion (MISN-0-17).

Output Skills (Knowledge):

- K1. Vocabulary: Newton's law of universal gravitation, universal gravitational constant.
- K2. Use Newton's law of universal gravitation, your correctly remembered value of g , and a solar system data table, to calculate the value of the universal gravitational constant G .

Output Skills (Problem Solving):

- S1. Calculate the vector gravitational force on one mass due to a given configuration of other masses.

Post-Options:

1. "The Cavendish Experiment" (MISN-0-100).
2. "Orbital Motion in an Inverse-Square-Law Force Field" (MISN-0-102).
3. "The Gravitational Field" (MISN-0-108).

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NEWTON'S LAW OF UNIVERSAL GRAVITATION

by
Paul M. Parker

1. Newton's Discovery of the Law

1a. Terrestrial Objects Accelerate Toward the Earth. If a compact object such as an apple is released from rest near the surface of the earth, it accelerates toward the center of the earth under the action of the gravitational force $F_g = mg$ where g is the local acceleration of gravity. The value of g near the surface of the earth is about 9.81 m/s^2 . If the object is not released from rest but with some initial velocity (not necessarily vertically up or down), the resulting motion is more complicated but still exhibits a constant acceleration of 9.81 m/s^2 directed towards the center of the earth.

1b. Newton Related Terrestrial and Celestial Motions. It was the genius of Newton which connected the gravitational acceleration of objects on the earth with the circular motion of the moon around the earth. The period T of this motion is about 27.3 days, the duration of one lunar cycle. The moon is sufficiently close to the earth for the earth-moon distance to be determinable by parallax relative to the positions of the background stars. This distance is the radius of the moon's orbit around the center of the earth and is $r = 3.84 \times 10^8 \text{ m}$, just about sixty times the radius of the earth ($R_e = 6.37 \times 10^6 \text{ m}$, see Fig. 1). Since for uniform circular motion

$$T = \frac{2\pi r}{v}, \quad (1)$$

the orbital speed v of the moon can be calculated knowing T and r , hence also its centripetal acceleration $a = v^2/r$. The associated centripetal force is then

$$F = ma = \frac{mv^2}{r}, \quad (2)$$

where m is the mass of the moon. Newton assumed that this centripetal force acting on the moon is provided by the gravitational attraction between the earth and the moon and therefore the associated centripetal acceleration experienced by the moon is the acceleration of gravity g_m that the earth exerts at the site of the moon. Carrying out this calculation it is found that earth's acceleration of gravity at a distance of sixty

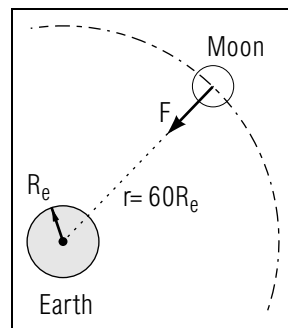


Figure 1. The uniform circular motion of the moon around the earth.

earth radii from the center of the earth is $g_m = 2.72 \times 10^{-3} \text{ m/s}^2$. Newton observed that this was just about $(1/3600)$ of the acceleration of gravity at the surface of the earth (one earth radius from the center of the earth).

1c. Newton Proposed an Inverse-Square Gravity Force. Since, going from one earth radius to a distance of sixty earth radii, the local acceleration of gravity is reduced by a factor of $(60)^2$, Newton concluded that the force of earth's gravity, mg , on an object of mass m is inversely proportional to the square of the distance of this object from the center of the earth and directly proportional to its mass m , i.e.,

$$F = \text{constant} \times \frac{m}{r^2}. \quad (3)$$

Since Newton's third law requires that two interacting objects exert mutual forces of equal magnitude and opposite direction, the above force would also have to meet the requirement

$$F = \text{constant} \times \frac{M}{r^2}, \quad (4)$$

where M is the mass of the earth, so it must meet the combined requirement:

$$F = \text{constant} \times \frac{Mm}{r^2}. \quad (5)$$

We give the proportionality constant the special symbol G , so Eq. (5) may be written as:

$$F = \frac{GMm}{r^2}. \quad (6)$$

This equation requires that in SI units the constant G has the dimensions of $\text{N m}^2/\text{kg}^2$. The constant G must not be confused with g , the local acceleration of gravity which has dimensions of m/s^2 . The quantity G is called the "gravitational constant."

2. Center-of-Mass Refinement

2a. The Earth and Moon Mutually Interact. As mentioned in the previous section, the earth-moon gravitational forces are mutual as required by Newton’s third law. Not only does the earth therefore influence the moon’s motion, but the moon also influences the earth’s motion. However, since the earth is more massive than the moon, the earth’s influence on the motion of the moon is the dominating influence and the moon’s influence on the motion of the earth is much smaller and can be neglected in all but very accurate calculations.

2b. The Earth and Moon Orbit Their Center-of-Mass. If the mutually-driven motions of the earth and moon are taken into account in a more careful way, it is found that the earth and the moon each orbit about their common two-body “center of mass.” However, this center of motion for each of them is located close enough to the earth’s center so that the resulting motion of the moon is not too different from our description based on placing the moon’s center of motion at the center of the earth.

The distance between the geometrical center of the earth and moon-earth system’s center-of-mass is about 73% of one earth radius or 1.2% of the moon’s orbital radius. The moon would orbit the earth’s center exactly if the earth had infinite mass.

3. Effects of Extended Objects

3a. Newton’s Law Assumes Point-Like Masses. We have discussed the earth and the moon as if they interacted gravitationally with their entire masses concentrated at their respective centers in a point-like manner. For two spherical objects like the earth and the moon separated by a distance large compared to their radii, this seems like a safe assumption. But how about an apple close to the surface of the earth? To the apple, does the attracting earth really appear equivalent to a point-like mass M located at the earth’s center? Newton was quite concerned about this point but finally succeeded in giving a rigorous mathematical proof that the answer to the above questions is yes. To complete the proof he needed the methods of calculus which he invented for that purpose.¹ The details of Newton’s proof show that one may treat a spherical object’s

¹The German mathematician G.W. Leibniz invented the calculus independently but did not apply it to the real world as Newton did. For the rest of his life Newton was unwilling to share the credit with Leibniz.

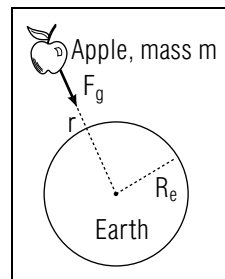


Figure 2. The weight of an object as a gravitational attraction.

mass as concentrated at its center if the object has uniform mass density or, more generally, if it is spherically symmetric (a mass density which depends only on the distance from the object’s center).

3b. Asymmetries Lead To Non-Uniform Acceleration. There is a slight variation in the value of the local acceleration of gravity over the surface of the earth because the earth is not a perfect sphere: it is flattened at the poles, and there are mountains, valleys, oceans, different types of soil and bedrock, etc. Also, there is a slight reduction in the local values, due to the rotation of the earth, at all points on the earth’s surface except the poles.

4. The Determination of G

4a. Gravitational Attraction is Equated to Weight. The gravitational constant G may be expressed in terms of the mass and radius of the earth, and the acceleration of gravity at the earth’s surface, by equating gravitational force with weight. Consider an apple of mass m at a distance r from the center of the earth (and therefore a distance $(r - R_e)$ above the surface of the earth). The force of gravity F on the apple is its local weight mg , and therefore the law of gravitation, Eq. (6), gives

$$mg = G \frac{Mm}{r^2}, \quad (7)$$

or, the local acceleration of gravity at the distance r from the center of the earth is

$$g = G \frac{M}{r^2}, \quad (r \geq R_e). \quad (8)$$

Applying this equation to a point on the earth’s surface where $r = R_e = 6.37 \times 10^6$ m and $g = g_e = 9.81$ m/s², we obtain

$$G = \frac{3.98 \times 10^{14} \text{ N m}^2 \text{ kg}^{-1}}{M}. \quad (9)$$

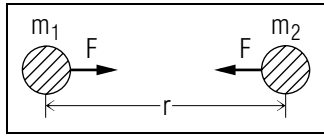


Figure 3. The gravitational interaction of arbitrary masses.

Thus, G can be calculated if the mass of the earth M is known.

4b. Newton Estimated the Earth's Mass. Newton did not know the mass of the earth, but he obtained a good estimate using the following reasoning. The mass of a spherical object of volume $(4\pi R_e^3/3)$ is given by

$$M = \left(\frac{4}{3}\pi R_e^3\right) \rho, \quad (10)$$

where ρ is the average density of the earth and R_e its known radius. To estimate the average density of the earth, Newton observed that the density of water is 1 g/cm^3 , and the density of a heavy metal like iron is about 7 g/cm^3 . A reasonable guess for the average density of the earth would thus be about 5 g/cm^3 . This approach gave Newton an estimate of G which was within 10% of the present accepted value of

$$G = 6.673 \times 10^{-11} \text{ N m}^2/\text{kg}^2. \quad (11)$$

The smallness of G requires that at least one of two gravitationally attracting objects be massive (like the earth) for the gravitational force to be readily apparent. Attractive forces between smaller objects, like two persons, are negligibly small (at least the gravitational kind) and are therefore completely masked by the gravitational attraction of the earth on each person.

4c. Cavendish Makes The First Direct Measurement of G .

The first laboratory measurement of G was made by the British scientist H. Cavendish in 1797, seventy years after the death of Newton. Note that the measurement of G can be used, along with Eq. (9), to determine the mass of the earth. More generally, a knowledge of G can be used to determine the mass of any planet if it has at least one moon whose period can be measured.

5. Universality of the Law

It is believed that Newton's law of gravitation is universal, i.e., it is operative everywhere in the universe. The value of G is assumed to be

the same everywhere and it is therefore called the "universal gravitational constant." There does not exist convincing evidence that the numerical value of G is changing in time on a geological or even a cosmic time scale. Thus universally any two point-like masses interact with forces of magnitude

$$F = G \frac{m_1 m_2}{r^2}. \quad (12)$$

Spherically symmetric objects can be treated as equivalent point-like masses. More irregularly shaped objects can also be treated by the law of universal gravitation with the help of integral calculus. Because of this universality, our earth-apple and earth-moon considerations can be transferred without modification to other planets and their moons, and as Newton convincingly demonstrated in 1666, to the motion of the planets of our solar system around the sun. Thus did Newton extend his terrestrial laws of mechanics to the heavens.

Acknowledgments

The author wishes to thank Professor James Linneman for several helpful suggestions. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **Newton's law of universal gravitation:** a law articulated by Sir Isaac Newton that describes the gravitational interaction between any two masses.
- **universal gravitational constant:** the constant of proportionality in Newton's law of universal gravitation.

PROBLEM SUPPLEMENT

Note: A symbol like [S-1] means that, if you have trouble getting the answer, you can get help from a sequence in this module's *Special Assistance Supplement*; in this case, from sequence [S-1].

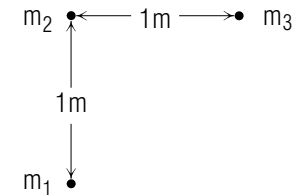
Problem 3 also occurs on this module's *Model Exam*.

PHYSICAL AND ORBITAL DATA FOR THE SOLAR SYSTEM*				
Body	Mean Radius ^c (meters)	Mass (kg)	Mean Orbital Radius (meters)	Orbital Period (seconds)
Sun	6.96×10^8	1.99×10^{30}	-----	-----
Mercury ^a	2.43×10^6	3.30×10^{23}	5.79×10^{10}	7.60×10^6
Venus ^a	6.06×10^6	4.87×10^{24}	1.08×10^{11}	1.94×10^7
Earth	6.37×10^6	5.98×10^{24}	1.50×10^{11}	3.16×10^7
Mars ^a	3.37×10^6	6.40×10^{23}	2.28×10^{11}	5.94×10^7
Jupiter	6.99×10^7	1.90×10^{27}	7.78×10^{11}	3.74×10^8
Saturn	5.84×10^7	5.69×10^{26}	1.43×10^{12}	9.30×10^8
Uranus	2.30×10^7	8.73×10^{25}	2.87×10^{12}	2.65×10^9
Neptune	2.22×10^7	1.03×10^{26}	4.50×10^{12}	5.20×10^9
Pluto	$< 3 \times 10^6$	$< 6 \times 10^{23}$	5.90×10^{12}	7.82×10^9
Moon ^b	1.74×10^6	7.35×10^{22}	3.84×10^8	2.36×10^6

*Data adapted from the Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, Her Majesty's Stationery Office, London (1961). Notes: ^aRadius obtained from radar observations, mass from space probe perturbations. ^bOrbital data are with respect to the earth. ^cRadius of a sphere of equal volume.

- Determine the ratio of the force of the sun on the moon to the force of the earth on the moon.
- Determine the force on you: (a) due to the earth; (b) due to the sun; and (c) due to the moon.
- Determine the resultant force on the sun due to the earth and Venus when:
 - they are on opposite sides of the sun;

- when their directions from the sun are at right angles. Sketch the situations.
- Find the net force on mass m_1 if all three masses shown below are 10.0 kg:



Express your answer in terms of the unit vectors $\hat{x} \rightarrow$ and $\hat{y} \uparrow$.

- Compare the gravitational attraction between a proton and an electron in a hydrogen atom to the attraction between the earth and the sun.

$$m_{\text{proton}} = 1.67 \times 10^{-27} \text{ kg}$$

$$m_{\text{electron}} = 9.11 \times 10^{-31} \text{ kg}$$

$$r_{e-p} = 0.53 \times 10^{-10} \text{ m}$$

Note: r_{e-p} above is the hydrogen atom electron-proton distance.

- Two people, each weighing 150 lb, are standing 3.0 feet apart. Calculate the gravitational force on each due to the other. (1 lb = 4.448 N, 1 ft = 0.3048 m).
- Use the value given for the universal gravitational constant G , the acceleration of gravity at the earth's surface, and the radius of the earth to calculate the mean density of the earth.
- (Only for those interested) You are a surveyor using a plumb line (a string with a weight at the lower end) near a mountain. Assume that the directions to the mountain and to the center of the earth are at right angles and that the mass of the mountain is 3×10^{14} kg. By how many degrees will your plumb line be off the vertical if the mountain is 3.0 km away? Sketch the situation.

Advice: Think of a line drawn from the top end of the plumb line to the center of the earth. Now think of a line drawn from the bottom end of the plumb line to the center of the earth. The center of the earth is so far away that the angle between these two lines is approximately

10^{-9} degrees! Thus you can consider them as parallel for the purposes of this problem.

Brief Answers:

1. $(F_s/F_e) = 2.18$ *Help: [S-6]*
2. To get numbers to compare to your own answers, put your weight into each of the lines below and, in the second and third lines, complete the indicated calculation.
 - a. $F_e = \text{your weight}$ *Help: [S-1]*
 $F_s = \text{your weight} \times (6.00 \times 10^{-4})$ *Help: [S-2]*
 $F_m = \text{your weight} \times (3.38 \times 10^{-6})$
 Note: the answers given above are *not* off by a factor of 10.
Help: [S-4]
3. a. $\begin{array}{c} e \\ \cdot \\ \leftarrow F_e \end{array} \quad \begin{array}{c} s \\ \cdot \\ \rightarrow F_v \end{array} \quad \begin{array}{c} v \\ \cdot \end{array}$
 $F_{net} = 2.01 \times 10^{22} \text{ N}$
 - b. $\begin{array}{c} v \\ \cdot \\ F_v \uparrow \\ \cdot \\ s \quad \vec{F}_e \quad \cdot \\ e \end{array}$
 $F_{net} = 6.57 \times 10^{22} \text{ N}$ *Help: [S-5]*
4. $\vec{F} = (2.36 \times 10^{-9} \text{ N})\hat{x} + (9.03 \times 10^{-9} \text{ N})\hat{y}$
5. $F(\text{electron-proton})/F(\text{earth-sun}) = 1.02 \times 10^{-69}$
6. $F = 8.3 \times 10^{-8} \text{ lb}$
7. $\rho = 5.51 \times 10^3 \text{ kg/m}^3$
8. 0.01°

SPECIAL ASSISTANCE SUPPLEMENT

S-1 (from PS-problem 2)

The (non-contact) gravitational force of the earth on you is what is commonly called your “weight.” It can be stated in pounds or in newtons.

S-2 (from PS-problem 2)

According to Eq. (12), the force varies linearly with mass and inversely with the square of the distance. *Help:* [S-3]

S-3 (from [S-2])

From Eq. (12):

$$\begin{aligned} F_s/F_e &= (G m_s m_{you}/r_{sy}^2)/(G m_e m_{you}/r_{ey}^2) \\ &= (m_s/m_e) \cdot (r_{ey}/r_{sy})^2 \\ &= \frac{1.99 \times 10^{30} \text{ kg}}{5.98 \times 10^{24} \text{ kg}} \cdot \left[\frac{6.37 \times 10^6 \text{ m}}{1.50 \times 10^{11} \text{ m}} \right]^2 \\ &= \dots \end{aligned}$$

Note 1: This ratio is a dimensionless number.

Note 2: The symbol r_{sy} is the distance from the sun to you, etc.

S-4 (from PS, problem 2)

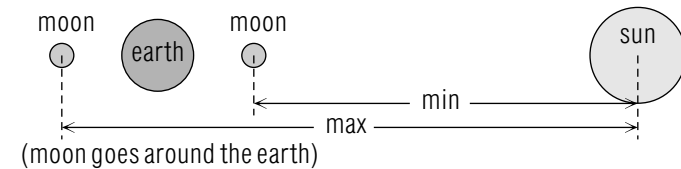
If you convert between mass and weight, do it properly. If you can't remember, look it up in the Index. Also, see [S-1].

S-5 (from PS, problem 3)

If you don't remember how to add vectors, look up vector addition in the Index.

S-6 (from PS, problem 1)

The distance of the moon from the sun is not given in the table of data so what should you use for that distance? To answer that question, use the distances of the moon from the earth and the earth from the sun to calculate the moon's distance from the sun when the moon, earth, and sun are all in a line and the moon is on the side of the earth toward the sun and again when the moon is on the side of the earth away from the sun (see the drawing below). These two distances are the minimum and maximum values for the moon-sun distance. Try each of those numbers in solving Problem 1 and see whether it matters which of the two numbers you use.



If you still can't get the correct numerical answer, see *Help:* [S-7]

S-7 (from PS, problem 1)

When people did not get the right numerical answer, we have found that they did not pay close attention to the wording of the problem and hence chose the wrong masses to put into the calculation.

MODEL EXAM

PHYSICAL AND ORBITAL DATA FOR THE SOLAR SYSTEM*				
Body	Mean Radius ^c (meters)	Mass (kg)	Mean Orbital Radius (meters)	Orbital Period (seconds)
Sun	6.96×10^8	1.99×10^{30}	-----	-----
Mercury ^a	2.43×10^6	3.30×10^{23}	5.79×10^{10}	7.60×10^6
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Pluto	$< 3 \times 10^6$	$< 6 \times 10^{23}$	5.90×10^{12}	7.82×10^9
Moon ^b	1.74×10^6	7.35×10^{22}	3.84×10^8	2.36×10^6

*Data adapted from the Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris and Nautical Almanac, Her Majesty's Stationery Office, London (1961). Notes: ^aRadius obtained from radar observations, mass from space probe perturbations. ^bOrbital data are with respect to the earth. ^cRadius of a sphere of equal volume.

1. See Output Skills K1-K2 in this module's *ID Sheet*. The exam may include one, both, or neither of those skills.
2. Determine the resultant force on the sun due to the earth and Venus when:
 - a. they are on opposite sides of the sun;
 - b. when their directions from the sun are at right angles.

Sketch the situations.

Brief Answers:

1. See this module's text.
2. See this module's *Problem Supplement*, Problem 3.

