

ANALYTIC FUNCTIONS

**Math
Physics**

ANALYTIC FUNCTIONS

by

R. D. Young, Dept. of Physics, Illinois State Univ.

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Author: R. D. Young, Dept. of Physics, Illinois State Univ.

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Input Skills:

1. Vocabulary: complex numbers, polar representation of complex numbers, complex algebra, exponential function, logarithmic function, complex conjugate, phase and modulus of complex number.
2. Unknown: assume (MISN-0-486).

Output Skills (Knowledge):

- K1. Write down from memory the definition or explanation of the following: polar representation of a complex number, modulus or magnitude of a complex number, phase of a complex number, complex function of a complex variable (C.V.), real part of a complex function, imaginary part of a complex function, complex conjugate, De Moivre's theorem, complex plane, logarithm function, branch of logarithm, principle branch, branch point, cut line or branch line, derivative of a function of C.V., Cauchy-Riemann (C.R.) conditions, analytic function, harmonic functions, exponential, hyperbolic and trigonometric functions.

Output Skills (Rule Application):

- R1. Perform simple calculations involving the elementary functions [i.e., $\sin(z)$, $\cos(z)$, $\sinh(z)$, $\cosh(z)$, $\exp(z)$.]
- R2. Test whether a function is analytic by using the Cauchy-Riemann conditions.
- R3. Construct the real (imaginary) part of an analytic function by using the C.R. conditions when given the imaginary (real) part of the function.

External Resources (Required):

1. G. Arfken, *Mathematical Methods for Physicist*, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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1. Introduction

This unit will introduce you to one of the most powerful mathematical techniques available to the scientist and engineer. You have already become acquainted with several physical problems giving rise to Laplace's equation. One of the chief implications of a function of a complex variable being differentiable is that the real and imaginary parts of these functions each satisfy Laplace's equation. Therefore, the theorems for analytic functions (functions of a complex variable differentiable in some open region) can be used when studying physical problems where Laplace's equation applies. This unit introduces the idea of a function of a complex variable as well as continuity and the derivative of such a function. Analytic functions are then studied.

2. Procedures

1. Review procedure. You should know the definition of complex numbers in rectangular form as well as complex algebra including addition, subtraction, multiplication, and division of complex numbers. If you do not, then see your instructor for further instructions.
2. Read pages 286-287 of Spiegel. Do not read the section on page 287 entitled Integrals and Cauchy's Theorem.
3. Read Sections 6.1 and 6.2 of Arfken. Do not read Section 6.3.
4. Underline in the texts or write out the definitions and explanations of the terms and concepts of Output Skill K1. Note: A discussion of the logarithm and the concept of branch is included in the Supplementary Notes which should be read now.
5. Write out the definitions of the trigonometric functions $\sin z$ and $\cos z$ and the hyperbolic functions $\sinh z$ and $\cosh z$ as given in problem 6.1.9 of Arfken in terms of power of series. The exponential function e^z is defined by:

$$e^z = \sum_{n=1}^{\infty} \frac{z^n}{n!}.$$

These definitions are asked for in Output Skill K1.

6. Read through Solved Problems 13.1, 13.2, 13.3, 13.5, and 13.7, pages 294-296, of Spiegel.
7. Solve these problems in Arfken: 6.1.9, 6.1.10, 6.1.14, and 6.1.15. All of these problems involve deriving various properties of the complex trigonometric and hyperbolic functions.
8. Solve these Supplementary Problems in Spiegel:
 - 13.55 (loci in complex plane)
 - 13.56 (regions in complex plane)
 - 13.57 (real and imaginary parts of complex functions)
 - 13.60 (derivative of a complex function)
 - 13.61 (Cauchy-Riemann conditions. Harmonic functions)
 - 13.62 (analyticity)
 - 13.64 (construction of an analytic function)
 - 13.65 (construction of an analytic function)
9. Solve this problem in Arfken on the construction of an analytic function using the C. R. conditions: 6.2.5.

3. Supplementary Notes

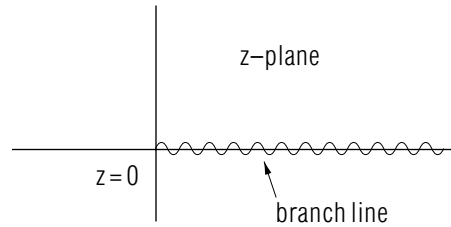
Branch Point and Branch Cut. Logarithm

Consider eq. (6.13b) of Arfken,

$$\ell n(z) = \ell n(r) + i(\theta + 2\pi n)$$

where the parameter n may be any integer. It is conventional to restrict the angle θ so that $0 < \theta < 2\pi$. Obviously ℓn is multi-valued. However, by restricting n to be a single integer and θ to lie in the range above, a single-valued function is obtained. Such a single-valued function obtained by restricting the phase of ℓn as above is called a *branch of the logarithm*. The principle branch of the logarithm (or any multi-valued function) is just the single-valued function obtained by restricting to the range above and choosing $n = 0$. The point $z = 0$ is the *branch point* of ℓn in the same sense as branch point is explained for z^2 in Spiegel, page 310. The

function \ln is made single-valued by arguing never to cross the line $\theta = 2\pi$ in the z -plane. Such a line of demarcation is called a *branch or cut line*. The branch point $z = 0$ lies at the point of termination of the branch line.



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