PHYSNET

MISN-0-466

LORENTZ TRANSFORMATIONS

Relativity

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

LORENTZ TRANSFORMATIONS

by

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Title: **Lorentz Transformations**

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Length: 2 hr; 12 pages

Input Skills:

1. Vocabulary: Mathematical group.
2. State the postulates of special relativity (MISN-0-465).

Output Skills (Knowledge):

K1. Define simultaneity (as in special relativity). Show that simultaneity depends on the motion of the observer.

K2. Outline the argument for establishing the form of the Lorentz transformation for a boost along a coordinate axis.

K3. Discuss in relation to relativity: (a) group property of Lorentz transformations; (b) uniqueness of the invariant speed c; (c) limiting nature of the speed c.

K4. State the Lorentz transformation for a boost along any coordinate axis in two forms (Lorentz factor γ, rapidity ϕ) and the relationship between the forms.

Output Skills (Rule Application):

R1. Given the coordinates of an event in one inertial frame, find its coordinates in another. Given the coordinates of an event in two inertial frames, find the relative velocity of the frames.

Output Skills (Problem Solving):

S1. Given information about the relative velocities of a series of inertial frames find the Lorentz transformation from any one of the frames to another.

External Resources (Required):

1. W. Rindler, *Essential Relativity* Van Nostrand, (1977).
2. A. P. French, *Special Relativity*, Norton (1968).

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New authors, reviewers and field testers are welcome.

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1. Introduction

In special relativity as in Newtonian mechanics it is possible to divide the discussion into two parts - kinematics and dynamics. Kinematics covers the mathematics of particle motion while dynamics is concerned with the forces that cause particles to move in the way that they do. The prescription for translating observations from one inertial frame to another is primarily a mathematical exercise and should be considered a part of kinematics. This prescription in Newtonian mechanics - the Galilean transformation - plays a rather secondary role and is quite often passed over and forgotten in introductory courses. However, its counterpart in special relativity - the Lorentz transformation - is essential to the entire discussion. Hence this unit is devoted to a development of the form of the Lorentz transformation in a special case.

2. Procedures

0. There are some things that need to be discussed and/or emphasized before jumping into the task at hand.

   a. First there are some assumptions that are almost always tacitly made in textbooks. They are perhaps obvious; but, until one realizes clearly that these assumptions have been made and not others, it is difficult to get a clear understanding of the beginning development of special relativity. For example, special relativity assumes the existence of inertial frames in which Newton's first law is valid (i.e. free particles travel in straight lines) and light obeys the law of rectilinear propagation. However, it makes no a priori assumption about the relative motion (which must be uniform in Newtonian physics) of two inertial frames. Furthermore, it makes no a priori assumption about the relationships between observations made in two different inertial frames.

      Just as in Newtonian physics, special relativity assumes space to be Euclidean, homogeneous and isotropic while time is assumed to be homogeneous (i.e. flow uniformly) when viewed from a single inertial frame. However, space and time are not assumed to be absolute in special relativity. Thus, special relativity allows for the numerical values of lengths and time intervals to be different when measured from different inertial frames although identical equipment is used.

   b. Besides the aforementioned assumptions, there is some terminology used in special relativity discussions that can be a little confusing. For example, it is common practice to speak of "standard clocks" and "standard measuring rods" as if they are something special. They are not! In fact, any clocks (or rods) are suitable provided that they are 1) identical in construction and 2) sufficiently precise for the discussion at hand.

      It is also common practice to speak of "such-and-such an event in reference frame S" as though that event only occurred in frame S or that it is only observable from frame S. You must realize from the beginning (and this cannot be over emphasized) that *all* events are observable by *all* observers independent of their state of motion. Thus "...in S" is short-hand for "...as observed by an observer who is at rest relative to frame S and who happens to be at the location of the event when it occurs."

   c. In the beginning study of relativity there is an occasional tendency to think that the consequences of special relativity are due simply to time delays resulting from the finite speed of light. Such time delays do produce interesting results but they are not at the heart of special relativity. In fact, such time delays exist even in Newtonian physics. In special relativity time delays only serve to complicate the discussion. To avoid such unnecessary complications, it is customary to assume the availability of an observer situated precisely at the location of the event(s) of interest. In the laboratory, of course, such observers are seldom, if ever, present and it becomes necessary to correct for delay times before investigating relativistic effects. Throughout this course (unless otherwise stated) the observer(s) will be assumed to be located at the position of the event(s) and no consideration of time delays need be made.

1. Read Rindler, sections 2.1 - 2.4. Omit second paragraph on p. 25

   Note: It is assumed that there is no problem concerning the simultaneity of events occurring at the same point in space. Only the simultaneity of events separated spatially is considered non-trivial.

   Comment: A little theorem is needed to make the discussion on pages 28 - 29 completely meaningful.

Theorem: If two parallel line segments, at rest relative to one another, are equal in one inertial frame, they are equal in all inertial frames.

Proof: Since the line segments are parallel and since space is homogeneous, one of the line segments can be moved into coincidence with the other one without its perceived length being altered. Now if their lengths are the same in one inertial frame (recall meaning of "…in S") their end- points will coincide in that frame. But the endpoints must then coincide in all frames. Now by homogeneity of space the line segment can be moved back to its original position without altering its length in any inertial frame. Hence the two lengths are equal in all inertial frames.

As a consequence of this theorem observers in both frames (S and S') in the discussion of section 2-4 of Rindler agree that M is the midpoint of PQ and M' is the midpoint of $P'Q'$. Hence M' and M must coincide in S when P and Q occur and finally M' must be to the right of M when the light signals arrive at M.

Note: The conclusion that Q happened before P in S' but simultaneously in S implies that although S' clocks are synchronized in S', and S clocks are synchronized in S, the S' clocks do not appear synchronized when viewed from S and vice versa.

Figure 1 shows the clocks when P and Q occurred, as seen from S.

Note that in S, $P'M' = M'Q'$ (in fact, $PM = P'M'$) but, clock $P' \neq$ clock $M' \neq$ clock Q', (i.e. S' clocks do not appear to be synchronized when viewed from S).

Figure 2 shows the clocks when Q occurred, as seen from S'.

Figure 3 shows the clocks when P occurred, as seen from S'.

Note that, in S', $PM = MQ$ (now, however, $PM < P'M'$ but the S clocks do not appear synchronized).

Study these figures very carefully!

▷ Exercise - Draw pictures of the clocks when M and M' coincide as seen from S'.

2. Read Rindler, sections 2.5 and 2.6.

   In the argument given by Rindler there are three main steps:

(1) Use of the definition-of inertial frames and an imposition of a finiteness requirement to establish the linearity of the equations. The linearity of the equations alone has interesting implications:

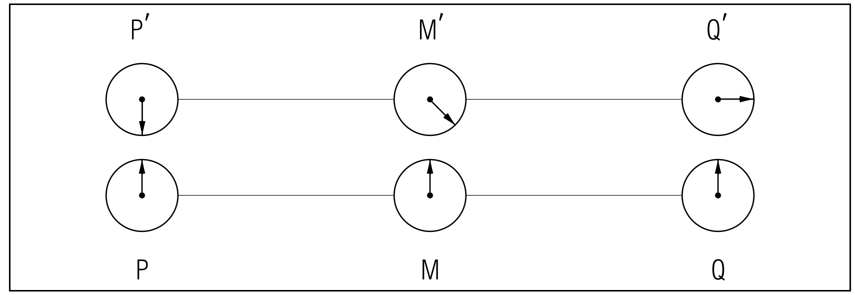

Figure 1. .

   (a) inertial frames have uniform translatory relative motion
   (b) the orientation of the axes of one frame as seen from another does not change in time. Once this is established Rindler specializes to the "standard configuration" case in which the coordinate axes coincide exactly at time $t = t' = 0$ and the relative motion is along the x-axis.

(2) Establishment of the "standard configuration" transformation equations for the spatial components perpendicular to the relative velocity using linearity and isotropy.

(3) Establishment of the "standard configuration" transformation equations for time and the space component parallel to the relative motion using linearity, isotropy and the second postulate of special relativity.

After studying Rindler see if you can reconstruct the argument on paper without reference to the text.

Comment - There are, of course, inertial frames that are not in standard configuration relative to one another. There are frames that are

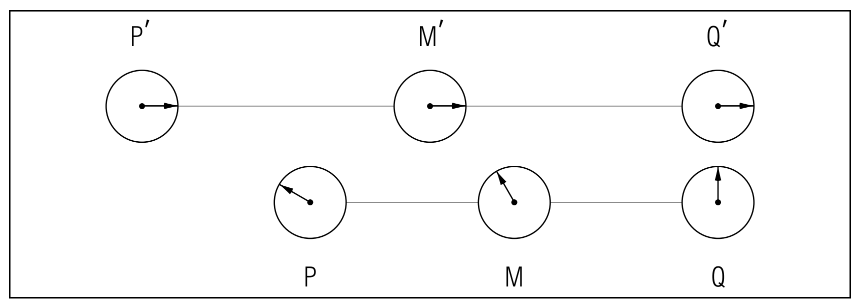

Figure 2. .

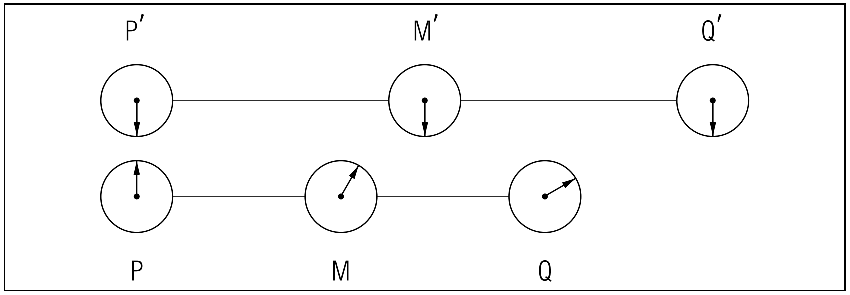

Figure 3. .

rotated (not rotating!), there are inertial frames whose origins don't coincide at $t = t' = 0$ and there are inertial frames whose origins never coincide. Fortunately, all Lorentz transformations can be expressed as a combination of four special types of Lorentz transformations:

1) *Boost* - connects inertial frames whose coordinate axes are parallel and whose origins coincide at $t = t' = 0$. (Note: the relative motion is not necessarily along the +x-axis. Rindler's "standard LT" is a special kind of boost.)
2) *Rotations* - connects inertial frames whose origins always coincide but whose axes are rotated (not rotating!) relative to one another.
3) *Space translations* - connects inertial frames with no relative motion whose coordinate axes are parallel and whose clocks agree but whose origins do not coincide.
4) *Time translations* - connects inertial frames with no relative motion whose axes coincide but whose clocks differ by a fixed amount.

This course is primarily concerned with boosts although one should be aware of the other three types of Lorentz (or Poincaŕe) transformations.

3. Read Rindler, section 2.7.

If you are not familiar with the concept of a group (mathematically) then it would be worthwhile to consult a math text. Convince yourself that the set of all Lorentz transformations forms a group.

4. Read Rindler, section 2.8.

Memorize equations 2.6, 2.7, 2.16 and 2.14 in Rindler. In equations 2.6 and 2.7 the quantity γ is called the *Lorentz factor* while in equations 2.16 and 2.14 the quantity ϕ is called the *rapidity.*

I find these equations much easier to remember using units in which $c = 1$.

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx)$$

Note that except for γ the first equation is just the Galilean transformation while the last equation can be obtained from the first by interchanging x and t everywhere.

In the rapidity form with $c = 1$ the boost becomes

$$x' = x\cosh\phi - t\sinh\phi$$
$$y' = y$$
$$z' = z$$
$$t' = t\cosh\phi - x\sinh\phi$$

Again, note the interchange of x and t in the first and last equations.

It is also very helpful to express these transformations in matrix form.

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

Note the symmetry of the transformation matrix.

Comment - Many books used to (and some still do) use a complex number notation. This is neither necessary nor useful. In fact, it can lead to severe complications especially when considering in more detail the group properties of Lorentz transformations. We will use real numbers only throughout our discussion of special relativity.

Comment - The use of $c = 1$ is extremely convenient and will be used throughout this course. If at any time (for numerical purposes) it is necessary to reinsert the c's this can easily be done from dimensional arguments alone.

▷ Exercise - Write down the Lorentz transformation for a boost at speed v along the z-axis.

▷ Exercise - An event occurs at the position (1 cm, 2 cm, 3 cm) at time $t = 10^{-10}$ sec in frame S. What are the coordinates (space and time) of this event in frame S' which is moving at a speed of $0.8c$ relative to S along the +y-axis? (assume S and S' are related by a boost.)

5. ▷ Exercise - Show that $v^2\gamma^2 = \gamma^2 - 1$

▷ Exercise:

a. Show that for v small ($v \ll 1$), $\gamma \approx 1 + 1/2v^2$
b. Show that for v large ($v \approx 1$), $\gamma \approx 1/\sqrt{2\delta}$, $\delta \equiv 1 - v$
c. Show that for $v = 0.99\ldots995$ (2n nines), $\gamma \approx 10^n$

▷ Exercise - Consider 3 inertial frames S, S' and S''. S' is boosted along x relative to S with speed v_1 while S'' is boosted along x relative to S' with speed v_2.

a. Find the transformation matrix from S to S''.
b. Show that

$$\gamma = \gamma_1\gamma_2(1 + v_1v_2)$$

$$v\gamma = \gamma_1\gamma_2(v_1 + v_2)$$

where $\gamma_1 = (1 - v_1^2)^{-1/2}$, etc.

c. Show that

$$v = \frac{v_1 + v_2}{1 + v_1v_2}$$

Note: This is done in a "slicker" way in Rindler, Section 2.8 , but this is probably more instructive.

▷ Exercise - Consider three inertial frames S, S' and S''. S' is boosted along x relative to S with speed v_1 while S'' is boosted along y relative to S' with speed v_2.

a. Find the transformation matrix from S to S''.
b. What is the speed of S'' relative to S? Hint: consider the motion of the origin of S''.

▷ Work problems 3-6 and 3-7 on p. 87 of French.

Acknowledgments

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