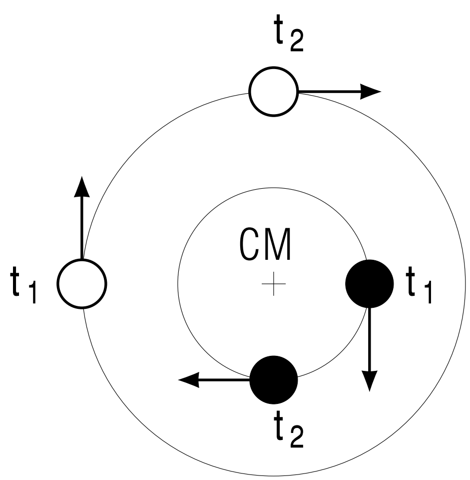
MISN-0-45

TWO BODY KINEMATICS AND DYNAMICS



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TWO BODY KINEMATICS AND DYNAMICS

by

Peter Signell

Title: **Two Body Kinematics and Dynamics**

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Input Skills:

1. Vocabulary: centrifugal potential energy; central forces; center of mass; frame of reference (MISN-0-58), (MISN-0-6), (MISN-0-11).

Output Skills (Knowledge):

K1. For an isolated two-body system, define these center of mass (CM) quantities in terms of single-particle quantities: position of CM ($\vec{R}$), total momentum ($\vec{P}$), total angular momentum ($\vec{L}$), total mass (M).

K2. Define these (CM-frame) quantities for an isolated two-body system: relative separation ($\vec{r}$), single-particle momentum ($\vec{p}$), total angular momentum ($\vec{\ell}$), and reduced mass (μ), in terms of single-particle quantities.

K3. Starting from single-particle equations, derive:

a. $\vec{P} = M\dot{\vec{R}} =$const.

b. $\vec{p} = \mu\dot{\vec{r}}$

c. $\vec{F} = \mu\ddot{\vec{r}}$

d. $F = \mu\ddot{r} - \ell^2/(\mu r^3)$ for central forces

e. $E_{tot} = \dfrac{M\dot{R}^2}{2} + \dfrac{L^2}{2MR^2} + \dfrac{\mu r^2}{2} + \dfrac{\ell^2}{2\mu r^2} + V(r)$ for central forces

f. $\vec{L}_{tot} = \vec{L} + \vec{\ell}$ where $\vec{L} =$ const., $\vec{\ell} =$ const.

K4. Show how the motions of two interacting particles are related to the equivalent one-particle solution. Use, for example, the earth and sun, earth and moon, earth and satellite, or a proton and an electron.

K5. For two masses connected by a spring with spring constant k: (a) write down the equations of motion for the masses; (b) transform to the CM system; (c) solve for the frequency of the system; (d) specialize the solution to $M_1 = M_2$; and (e) specialize the solution to $M_2 \gg M_1$.

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1. Introduction

1a. Two-Body Systems. Special techniques have been developed for predicting the future motions of two objects which interact solely with each other. A few examples of such binary systems are: the earth and its moon, binary stars, the hydrogen atom (an electron-proton system), diatomic molecules, and the scattering of the elementary particles of high energy physics. Of course, in each of these cases the interacting pair is not completely isolated from interaction with all other objects in the universe. However, for most of them, and under most conditions, a great deal of insight and useful solutions can be obtained by solving the system as though it were an isolated one. In some cases the interactions from other objects are too small to be significant and so can be validly neglected.

1b. Incremental Solutions of Particle Dynamics. Incremental solutions of particle dynamics rely on the application of Newton's second law of motion to each object separately. That is, given the mass of an object and the net (vector) force on it, the (vector) acceleration of the object can be computed. If one desires the trajectory the object will follow, a knowledge of position and speed at some initial instant is needed. Then the computed acceleration can be used to compute add-ons to the initial velocity. The computed velocities can, in turn, be used to compute add-ons to the initial position. In general, the force on the object will vary according to the object's position in space. This means that the calculation of a trajectory add-on, resulting in a new position at a new time, results in a new force with which to evaluate the next trajectory add-on. Thus the trajectory can be traced out, step by step.

1c. Connected Vector Equations, Binary Systems. For the case of two interacting particles, the general approach to particle dynamics needs to be supplemented, both for ease of solution and for gaining insight. Straightforward application of Newton's second law to one particle requires knowledge of the force on that particle. Since the force on it is produced by the other particle, the magnitude and direction of that force is dependent on the location of the other particle. One can calculate the velocity and trajectory add-ons of the first particle for a small

time interval, producing a new first-particle position at the incremented time. However, in the computation of the force for the next time interval, one must take into account the new position of the other particle. It has moved, since, by the third law, it has experienced a (vector) force equal but opposite to that experienced by the first particle. To summarize: There is a (vector) second law of motion for each of the interacting particles, but these equations are connected by the common-but-opposite force which produces the inter-particle interaction. The two (vector) equations of motion must be solved simultaneously.

1d. Six Equations, Twelve Initial Conditions. For binary systems, straightforward application of the general method of particle dynamics results in six coupled position-component equations whose simultaneous solutions must be consistent with twelve specified initial conditions. The six coupled equations result from, for example, taking the Cartesian components of the two three-dimensional vector equations of motion. Similarly, the initial three-dimensional positions and velocities of the two particles are specified through a total of twelve initial condition numbers.

2. The CM Frame

2a. CM-Frame Simplification of the Equations of Motion. The general method used for simplifying the six equations of motion for binaries is that of "going to the center of mass (CM) frame"; that is, one solves the problem as seen from the frame of reference in which the binary center-of-mass is at rest. This frame is inertial, so the six equations of motion can be solved there and then be easily transformed to any other inertial frame of interest. The advantage of the CM frame is that the six equations of motion can there be rearranged into two decoupled sets of three equations. One of these two sets can be solved trivially, and the other set can be reduced to a single equation. The usual single-particle techniques of solution can then be applied to it.

2b. Equal But Opposite Momenta. In a frame of reference in which a binary system's center-of-mass (CM) is at rest, not moving, the individual (vector) momenta of the two particles turn out to be equal but opposite. To see this, consider the earth-and-satellite system shown in Fig. 1 at some instant of time. The vector to the system's CM is given

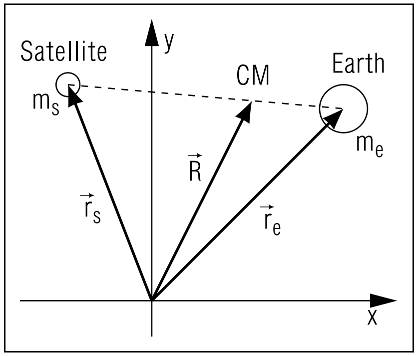

Figure 1. The CM is on the line joining the two masses.

by:[1]

$$\vec{R} = \frac{m_e \vec{r}_e + m_s \vec{r}_s}{m_e + m_s}. \tag{1}$$

This point is on the line joining the masses, and its velocity is:

$$\frac{m_e \vec{v}_e + m_s \vec{v}_s}{m_e + m_s}, \tag{2}$$

as follows directly from Eq. (1) by differentiation. Now imagine moving along at the same velocity as the CM, $\vec{V}_{CM}$. In this moving "CM" frame of reference, the CM itself will be observed to be at rest and so its velocity is zero:[2]

$$\vec{V}' = \frac{m_e \vec{v}_e^{\,\prime} + m_s \vec{v}_s^{\,\prime}}{m_e + m_s} = 0. \tag{3}$$

This means that:

$$m_e \vec{v}_e^{\,\prime} = -m_s \vec{v}_s^{\,\prime},$$

which is usually written:

$$\vec{p}_e^{\,\prime} = -\vec{p}_s^{\,\prime}.$$

Thus the CM-frame momenta are equal but opposite.

2c. Circular Motion in a Binary CM Frame. The motions of a binary system are particularly simple in the system's CM frame of reference. For example, consider the case where one of the two particles describes a circle around the CM, as in Fig. 2 (note in the figure that the total momentum of the two particles is always zero). The other particle must also describe a circle around the CM because the ratio of the two particles' distances from the CM is fixed at the ratio of their masses.[3]

[1]See "Static Equilibrium, Center of Mass" (MISN-0-6).
[2]Quantities measured in the CM frame are indicated by primes.
[3]That is, for an earth-satellite system, $r_s/r_e = m_e/m_s$. See Eq (1), $\vec{R} = 0$.

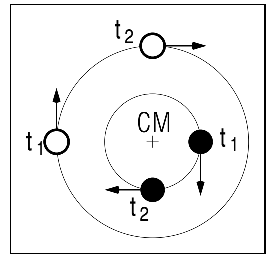

Figure 2. Two circularly orbiting particles at several times.

Thus, for example, the earth and moon traverse concentric circular orbits about the earth-moon CM.[4]

2d. General Equation of Motion. The equation of motion of each particle in a binary system is particularly simple in the CM frame. For example, consider the earth-satellite system shown in Fig. 3. If $\vec{r}$ denotes the relative position vector to the satellite, then the earth and satellite CM position vectors are given in terms of the relative position by:[5]

$$\begin{aligned} \vec{r}_s &= \frac{m_e}{m_e + m_s}\vec{r}, \\ \vec{r}_e &= -\frac{m_s}{m_e + m_s}\vec{r}. \end{aligned}$$

Note that:

$$\vec{r} = \vec{r}_s - \vec{r}_e$$

and

$$r_s/r_e = m_e/m_s.$$

[4]This point is not at the center of the earth, but is between its center and its surface.
[5]For a further discussion of CM see "Static Equilibrium, Centers of Force, Gravity and Mass" (MISN-0-6).

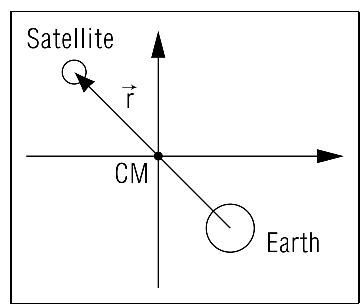

Figure 3. An earth-satellite binary system at some instant of time ($\vec{r}$ is the relative position vector).

Newton's second law of motion can be applied to each particle. Then, using a conveniently defined quantity called the *reduced mass*, μ,

$$\mu \equiv \frac{m_e m_s}{m_e + m_s},$$

we get:[6]

$$\vec{F}_s = \mu \ddot{\vec{r}} \qquad (4)$$
$$\vec{F}_e = -\mu \ddot{\vec{r}}.$$

This also demonstrates Newton's third law of motion. Equation (4) is mathematically identical to the equation one would get for a single fictitious particle of mass μ at a distance r from the origin. Of course, no such single particle is at that radius but the equation allows us to use the usual single-particle techniques for solution.

2e. Relative Momentum. Although the total momentum is always zero in any system's CM frame, the individual particle momenta generally are not. Using an earth-satellite system as an example:

$$\vec{p}_s = m_s \dot{\vec{r}}_s = \mu \dot{\vec{r}} \equiv \vec{p}$$

and

$$\vec{p}_e = -\vec{p}.$$

Thus the momentum of the satellite is equivalent to that of a fictitious particle of mass μ at a radius r from the origin. We call this the CM-frame "relative" momentum $\vec{p}$.

2f. Kinetic Energy and Angular Momentum. Applying the single-particle definitions of kinetic energy and angular momentum separately to each object in an earth-satellite binary system, the total CM frame quantities can be derived directly as:

$$\vec{\ell} \equiv \vec{\ell}_s + \vec{\ell}_e = \vec{r} \times \vec{p}$$

and

$$E_k \equiv E_{k,s} + E_{k,e} = \frac{1}{2}\mu|\dot{\vec{r}}|^2 = \frac{p^2}{2}. \qquad (5)$$

Notice the perfect equivalent-particle analogy in both cases. For central forces, Eq. (5) can be written as:[7]

$$E_k = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2}\ .$$

[6]Each dot above a symbol indicates a time derivative. Thus, $\ddot{\vec{r}} \equiv \dfrac{d^2\vec{r}}{dt^2}$.

[7]See "Derivation of the Constants of the Motion for Central Forces" (MISN-0-58).

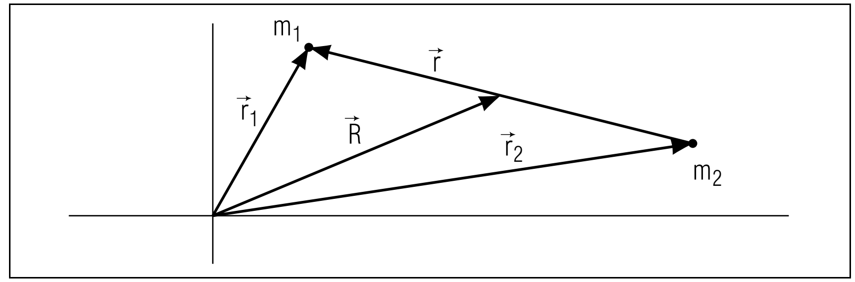

Figure 4. Illustration of the CM and relative vectors.

3. Formal Derivation

3a. The Equations of Motion. For a general binary System, the six coupled equations of motion can be put into a more tractable form by converting to CM and relative coordinates. For two interacting objects with position vectors and masses as denoted in Fig. 4, the CM and relative coordinates are defined by:

$$\vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \qquad (6)$$

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2. \qquad (7)$$

The velocity of the CM is found from Eq. (6) to be:

$$\vec{V} \equiv \dot{\vec{R}} = \frac{\vec{p}_1 + \vec{p}_2}{m_1 + m_2}.$$

The numerator is just the total momentum, which is conserved because there is assumed to be no external force on the binary system. Thus the system's CM always traverses a straight line at constant speed, no matter what the individual particles are doing. The latter are governed by the individual-particle equations of motion which are straightforwardly found to be:[8]

$$\vec{F}_1 = \mu \ddot{\vec{r}}; \qquad \vec{F}_2 = -\mu \ddot{\vec{r}},$$

where the system's reduced mass is:

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}.$$

[8]Each overhead dot represents a time derivative: $\ddot{\vec{r}} \equiv d^2\vec{r}/dt^2$.

3b. Momentum, Angular Momentum, and Kinetic Energy. The equations for various kinematical quantities can be determined in the CM relative notation by straightforward application of the single-particle definitions of those quantities. For example, the total momentum is

$$\vec{p}^{\,tot} = \vec{p}_1 + \vec{p}_2 = M\vec{V},$$

where M is the system's total mass. Similarly one finds the total angular momentum to be:

$$\vec{L}^{tot} = \vec{R} \times \vec{P} + \vec{r} \times \vec{p} \equiv \vec{L} + \vec{\ell};$$

where:[9]

$$\vec{p} \equiv \mu\dot{\vec{r}} \equiv \mu\vec{v}.$$

Note that $\vec{p}$ is the first particle's CM momentum and the negative of the second's. The kinetic energy of the system is found to be:[10]

$$E_k^{tot} = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 = \frac{P^2}{2M} + \frac{p^2}{2\mu} = \frac{1}{2}M\dot{R}^2 + \frac{L^2}{2MR^2} + \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2}.$$

In each of the above total-system variables, there is a CM part plus a CM-frame part. This is easily seen by noting that $\vec{P} = 0$ in the CM frame, while $\vec{p} = 0$ if there are no motions in the CM frame. Interestingly, the CM-frame terms are just those of a fictitious particle of mass μ and position $\vec{r}$ with respect to the CM as origin.[11]

4. A Problem: Two Masses and a Spring

4a. Statement of the problem.

[9]For two equal mass particles, p is half the particles' momentum difference:

$$\vec{p} = \frac{1}{2}\left(\vec{p}_1 - \vec{p}_2\right).$$

[10]See "Derivation of the Constants of the Motion for Central Forces" (MISN-O-58).

[11]Here is another interesting expression used in wave theory:

$$\vec{p}_1 \cdot \vec{r}_1 + \vec{p}_2 \cdot \vec{r}_2 = \vec{P} \cdot \vec{R} + \vec{p} \cdot \vec{r}.$$

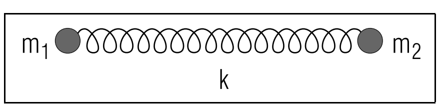

Figure 5. Two masses connected by a spring.

1. Consider two masses, M_1 and M_2, connected by a spring with spring constant k, as shown in Fig. 5.. Write down the equations of motion for the masses, then transform to the CM system and solve for the frequency of the system. Finally, specialize the general solution to these two cases:
   a. $M_1 = M_2$.
   b. $M_2 \gg M_1$, as in the case of an ordinary mass connected to the earth by a spring (let $M_1/M_2 \to 0$).

4b. Solution.

$$F_1 = -k(x_1 - x_2) = m_1\ddot{x}_1 \tag{8}$$

$$F_2 = -k(x_2 - x_1) = m_2\ddot{x}_2 \tag{9}$$

In the CM frame, with the CM at the origin, define: $x \equiv x_1 - x_2$. Then:

$$x_1 = \frac{m_2}{m_1 + m_2}\,x$$

and

$$x_2 = -\frac{m_1}{m_1 + m_2}\,x$$

Then Eqs. (8) and (9) become:

$$-kx = m_r\ddot{x} \tag{10}$$

$$kx = -m_r\ddot{x} \tag{11}$$

where $m_r \equiv m_1 m_2/(m_1 + m_2)$ is the "reduced mass." Note that Eqs. (10) and (11) are the same equation. It describes a spring with a mass m_r on one end and with the other end fixed (good for all mass-ratio cases), so the solution for the frequency of oscillations is:[12]

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m_r}}$$

a. if $m_1 = m_2$, then $m_r = m_1/2 = m_2/2$, so

$$\nu = \frac{1}{2\pi}\sqrt{\frac{2k}{m_1}}$$

b. if $m_2 \to \infty$ (which fixes the #2 end), then $m_r \to m_1$ and

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m_1}},$$

a lower frequency than for $m_1 = m_2$.

[12]See "Simple Harmonic Motion" (MISN-0-25).

Acknowledgments

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