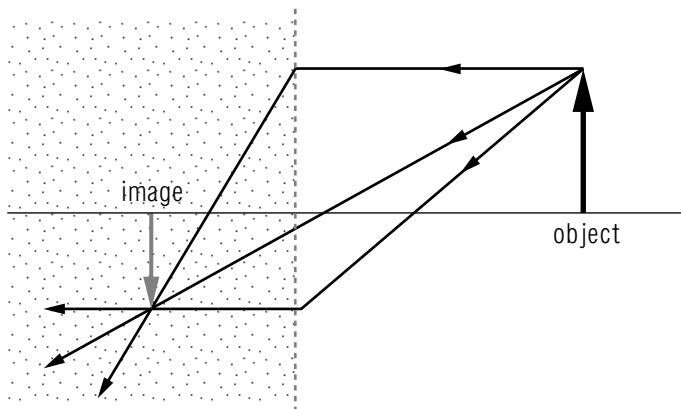


# REFRACTION AT SPHERICAL SURFACES



## REFRACTION AT SPHERICAL SURFACES

by  
M. Brandl

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**Input Skills:**

1. Vocabulary: refracted ray, index of refraction, Snell's law (MISN-0-220), optical image, optical object, focal point, focal length, center of curvature, principal axis, principal rays, central ray, focal ray, parallel ray, real image, virtual image, magnification, paraxial ray (MISN-0-221).
2. Use Snell's law to trace rays entering, traversing, and leaving slabs and prisms (MISN-0-220). geometrical optics (MISN-0-221).

**Output Skills (Knowledge):**

- K1. Vocabulary: Descartes' (refraction) formula, sign convention (optics), concave, convex, converging system, diverging system, image focus, object focus.
- K2. Given Descartes' formula, explain all symbols.

**Output Skills (Problem Solving):**

- S1. Solve problems, graphically or using Descartes' formula as required, for spherical refracting surfaces. Determine the location of the focal points, determine whether the image is real or virtual, upright or inverted, and determine the image size.

**Post-Options:**

1. "Thin Spherical Lenses" (MISN-0-223).
2. "Optical Instruments" (MISN-0-224).

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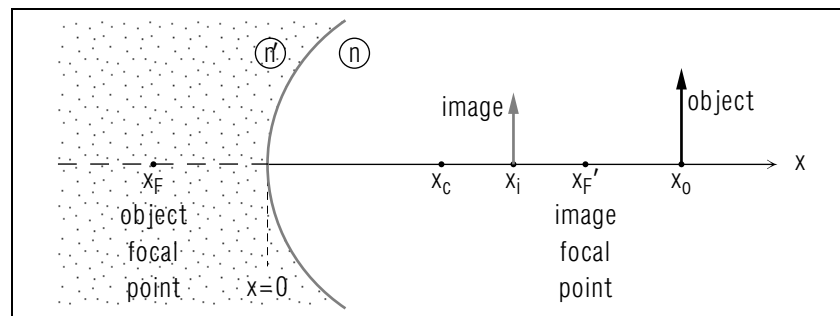
## 1. Introduction

**1a. Image Formation.** We can deal with images formed by refraction in much the same way we deal with images formed by reflection. That is, we form the image point by point, by taking two rays from each point on the object, following them through the refracting surface, and placing the image point where they either intersect (for a real image) or appear to have intersected (for a virtual image).

**1b. The Sign Convention.** One of the problems in geometrical optics requiring careful attention is the matter of sign conventions. A sign convention is a set of rules telling when the object or image distances, focal lengths, radii of curvature of surfaces, and so forth, are positive or negative. We shall use a notation involving positions of points relative to a coordinate origin.<sup>1</sup> Whether a quantity to be used is positive or negative is then obvious—it depends solely upon which side of the origin the point in question lies.

**1c. Our Notation.** A typical situation that illustrates our notation is shown in Fig. 1. The  $x$ -axis coincides with the principal axis of the system, defined as the line passing through a reference point on the object and the center of curvature of the surface. The  $x$ -axis origin is the point at which the refracting surface intersects the axis. In the case shown in the figure,  $x_o$  (object position),  $x_{F'}$  (image focus position),  $x_i$  (image position), and  $x_C$  (position of center of curvature of surface) are all positive, while the  $x_F$  (object focus position) shown has a negative value. In general, however, any of these quantities can have either positive or negative values. In the case shown, the medium to the right of the origin has index of refraction  $n$ ; to the left,  $n'$ .

<sup>1</sup>See “Optical Instruments” (MISN-0-224) for application of the principles derived in this module to the properties of lenses intended for optical instruments and the correction of eye defects.



**Figure 1.** Illustration of the notation we use, shown for refraction at a spherical surface.

## 2. Descartes' Formula

**2a. Method of Derivation.** For refraction at a single surface, Descartes' formula gives image position in terms of object position, plus the two indexes of refraction and the surface's radius of curvature (see Fig. 1). To derive the formula we must trace two rays from the object to the image. One of these can be a ray directed through the center of curvature ( $x_C$  in Fig. 1). That ray will strike the refracting surface at right angles and pass straight through without being refracted. Then we need only trace one other ray; the point at which that ray intersects the first ray will be the image position  $x_i$ . Note that the second ray leaves the object point  $x_o$  heading in a different direction from the first ray. Therefore it can only intersect the first ray if its direction is changed at the point where it crosses the refracting surface. That change of direction is governed by Snell's law:

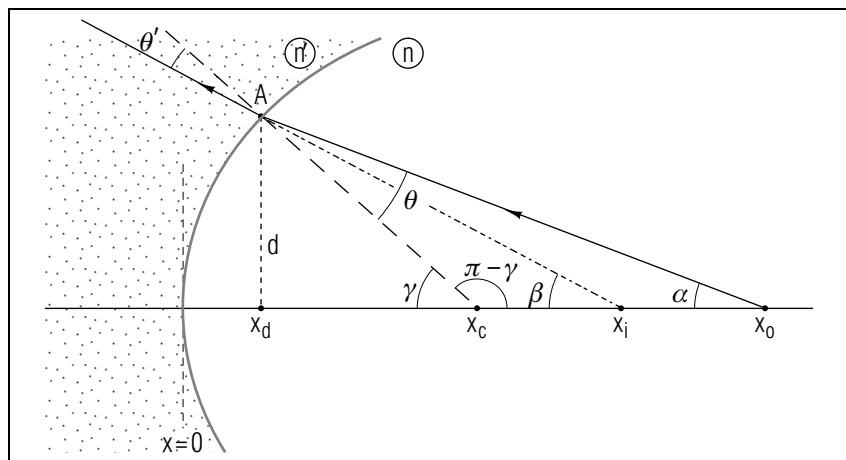
$$n \sin \theta = n' \sin \theta'. \quad (1)$$

Here  $n$  and  $\theta$  are for the medium to the right of the origin while  $n'$  and  $\theta'$  are for the one on the left (see Fig. 2). Then a little trigonometry, along with the “paraxial ray” approximation produces:<sup>2</sup>

$$\frac{n}{x_o} - \frac{n'}{x_i} \approx \frac{n - n'}{x_C} \quad (\text{paraxial rays}) \quad (2)$$

This is Descartes' formula for the approximate refraction produced on paraxial rays by a single spherical surface.

<sup>2</sup>See Appendix A for the gory details.



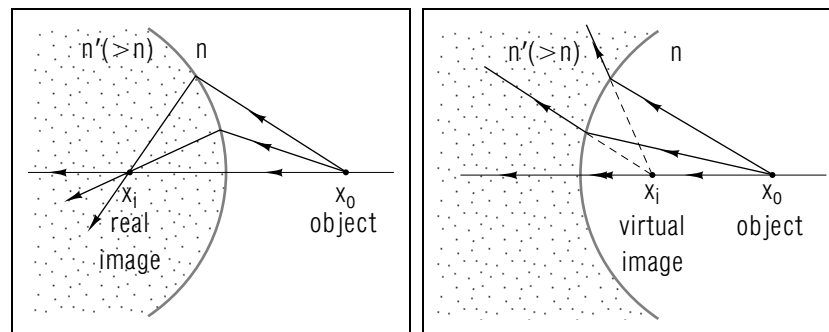
**Figure 2.** Angles used in the derivation of Descartes' formula.

**2b. Concave and Convex Surfaces.** Descartes' formula, Eq. (2), is applicable regardless of whether any of the object, image, and center of curvature positions are to the left or right of the origin. However, we will usually place the object to the right of the refracting surface, making  $x_o$  positive. Here are some definitions of surfaces:

A "convex" surface is one that bulges out in the middle toward the object, so that its center of curvature is on the opposite side of the surface from the object. Therefore a surface is "convex" if  $x_o$  and  $x_c$  have opposite signs. This usually indicates a negative value for  $x_c$ .

A "concave" surface has its center of curvature on the same side of the surface as the object. A surface is "concave" if  $x_o$  and  $x_c$  have the same sign. This usually indicates a positive  $x_c$ .

**2c. Real and Virtual Images.** A "real" image point is a point at which the rays from the object actually do intersect. Since all the rays on the object's side of the surface are diverging from the object, they can only converge on the opposite side of the surface after having been refracted. Therefore, an image is "real" if  $x_o$  and  $x_i$  have opposite signs. This usually indicates a negative  $x_i$  (see Fig. 3). A virtual image is said to be "at" that point from which the rays from the object appear to be diverging after having been refracted. The virtual image will appear to



**Figure 3.** Real image formation;  $n' > n$ .

**Figure 4.** Illustration of virtual image formation,  $n' > n$ .

be on the same side of the surface as the object. Therefore, an image is virtual if  $x_o$  and  $x_i$  have the same sign (in usual practice,  $x_i$  is then positive). See Figure 4. The distinction between real and virtual images is the same for images produced by refraction as it is for images produced by reflection. A real image can be seen on a screen, while a virtual image can be seen only if the eye can intercept the rays that appear to be diverging from the virtual "image point."

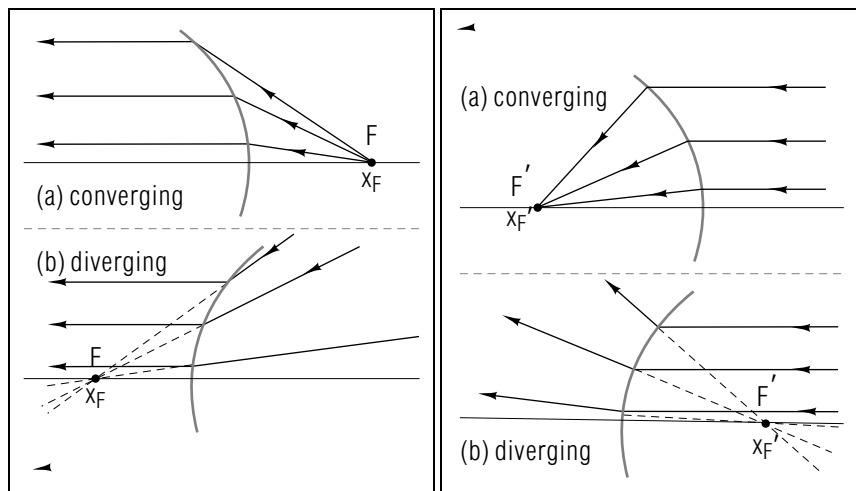
### 3. Focal Points

**3a. Definitions and Positions of Focal Points.** There are two particular space points that are quite useful in drawing ray diagrams.<sup>3</sup> These are the "focal points" or "foci" of the system. The "object focus" or "first focal point,"  $F$ , is that object position which produces an image at infinity. Any ray passing through  $F$  will leave the refracting surface traveling parallel to the principal axis (Fig. 5). The position of  $F$  can be obtained by putting the image at infinity,  $x_i = \pm\infty$ , in Descartes' formula:

$$x_F = \frac{nx_C}{n - n'}. \quad (3)$$

The "image focus" or "second focal point,"  $F'$ , is the image position that is produced by an object at infinity. Any ray coming in parallel to the principal axis will pass through  $F'$  after being refracted (Fig. 6).

<sup>3</sup>See "Spherical Mirrors" MISN-0-221.



**Figure 5.** Two types of object focal points.

The position of  $F'$  is obtained by setting  $x_o = \infty$  in Descartes' formula:

$$x_{F'} = -\frac{n'x_C}{n - n'}. \quad (4)$$

**3b. Converging and Diverging Systems.** You will notice that Figs. 5 and 6 both show two situations, one labeled “converging” and one labeled “diverging.” The definition of these two terms can be inferred from Fig. 6. A “converging” system is one which causes rays that enter traveling parallel to the axis to converge toward the axis after refraction. A “diverging” system is one which causes rays that enter traveling parallel to the axis to diverge away from the axis after refraction. Examination of Figs. 5 and 6 shows that a system is converging if the object focus lies on the object side (in usual practice,  $x_F$  positive), while the system is diverging if the object focus lies on the image side (in usual practice,  $x_F$  negative). Eqs. (3) and (4) combine to give

$$x_{F'} = -\frac{n'}{n}x_F, \quad (5)$$

so the object focus and image focus are always on opposite sides of the surface.

**3c. Descartes' Formula With  $x_F$ .** We can combine Eqs. (2), (3) and (4) to give Descartes' formula in terms of the position of the focal points:

$$\frac{n}{x_o} - \frac{n'}{x_i} = \frac{n}{x_F} = -\frac{n'}{x_{F'}}. \quad (6)$$

This is a very useful form of the law.

**3d. Image Positions.** Now that we know the properties of the focal points and can find their positions, we can use those properties to find the positions and sizes of images by drawing geometric ray diagrams. We need only use two points on the object to form an image, since the positions of the two corresponding image points will tell us where the image is, how big it is, and whether it is upright or inverted. The first point chosen is usually considered to be the “head” of the object, at a height  $h$  above the principal axis. The position of its corresponding image point can be found using any two of the three principal rays of the system (see Fig. 7).

The first principal ray, the parallel ray  $PR$ , starts at the object point and travels parallel to the axis. After refraction, it passes through the image focus  $F'$ .

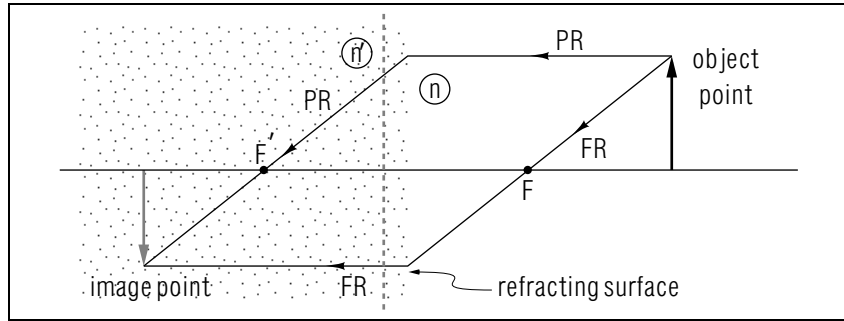
The second principal ray, the focal ray  $FR$ , starts at the object point and passes through the object focus  $F$ . After refraction it travels parallel to the axis.

The third principal ray, the central ray  $CR$ , starts at the object point and passes through the center of curvature of the refracting surface. Since it is incident along a surface normal, it is not refracted.

The image point is the point at which the refracted rays intersect. We really need just two of the principal rays to locate an image point. Usually only the parallel ray and focal ray are used.

Notice that we have replaced the refracting surface with a flat plane (perpendicular to the paper) in Fig. 7. Precisely this replacement is implied by the small-angle approximations used in deriving Descartes' formula.<sup>4</sup> In practice, one first determines the focal point positions numerically from Descartes' formula, then uses these values to draw ray diagrams such as in Fig. 7.

<sup>4</sup>A proof is given for the similar case of spherical mirrors in MISN-0-221.



**Figure 7.** Ray diagram for typical converging system.

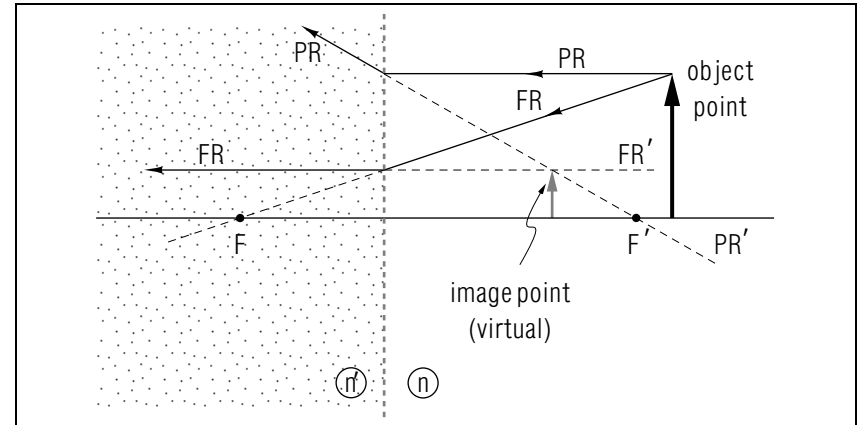
We can now find a second image point by just picking another object point and tracing principal rays from it. Suppose we were to pick an object point halfway between the head and the tail of the object. The ray diagram locating the image of this half-way point is similar to the one for the head of the object, the only difference being that it has only one-half the size in the vertical direction. The image point is thus halfway between the head and tail of the image. Any object points having the same  $x$ -coordinate will give rise to image points all of which have the same  $x$ -coordinate. If the object is perpendicular to the principal axis, then so is the image.

Fig. 7 is a ray diagram for a converging system. The ray diagram for a diverging system, Fig. 8, looks a bit more complicated, but is drawn using precisely the same rules. The central ray (CR) is the only one that actually passes through the image point, since only the virtual extensions ( $PR'$  and  $FR'$ ) of the parallel and focal rays ( $PR$  and  $FR$ ) pass through that point. The outgoing parallel and focal rays do not intersect at the image point.

**3e. Image Height and Magnification.** We can use a simple ray diagram and trigonometry to find the size of an image,  $h'$ , in terms of the position and size of the object,  $h$ , and the position of the object focal point (see Appendix B):

$$h' = - \left( \frac{x_F}{x_o - x_F} \right) h. \quad (7)$$

We can also use the fact that the central ray goes straight from the tip of the object, through the center of curvature of the surface to the tip of the image, all without being deflected. This gives us an expression in terms



**Figure 8.** Ray diagram for a diverging system.

of the positions of the object, image, and center of curvature:

$$h' = \left( \frac{x_i - x_C}{x_o - x_C} \right) h. \quad (8)$$

Finally, we can get the most easily remembered expression for the height of the image by taking Descartes' formula in the form given in Eq. (6), using it to express the positions of the focal points in terms of the object and image positions, and then substituting one of those expressions into Eq. (7), giving:

$$h' = \left( \frac{nx_i}{n'x_o} \right) h \quad (9)$$

The magnification  $M$  is defined to be the ratio of image size to object size:

$$M \equiv \frac{h'}{h}.$$

Note that the magnification is negative in Fig. 7, positive in Fig. 8.

## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

## Glossary

- **sign convention (optics):** a set of rules telling when the various distances that occur (positions of object, image, center of curvature, focal length) are to be considered positive and when they are to be considered negative.
- **concave:** in reference to the surface of a mirror or a lens, it indicates that the surface bends toward the positive  $x$ -direction (the direction toward a real object) as one moves up the surface away from the principal axis.
- **converging system:** a system which causes rays that enter the system, traveling parallel to the axis, to converge toward the axis after refraction.
- **convex:** in reference to the surface of a mirror or a lens, it indicates that the surface bends away from the positive  $x$ -direction (the direction toward a real object) as one moves up the surface away from the principal axis.
- **diverging system:** a system which causes rays that enter traveling parallel to the axis to diverge away from the axis after refraction.
- **image focus:** the image position produced by an object at infinity.
- **object focus:** the object position that produces an image at infinity.

### A. Derivation of Descartes' Formula

(only for those interested)

The geometrical layout of the derivation is illustrated in Fig. 2 of the module text. Now examine the two triangles  $(Ax_i x_C)$  and  $(Ax_o x_C)$ . Since the sum of the interior angles in a triangle is  $\pi$  radians, we have:

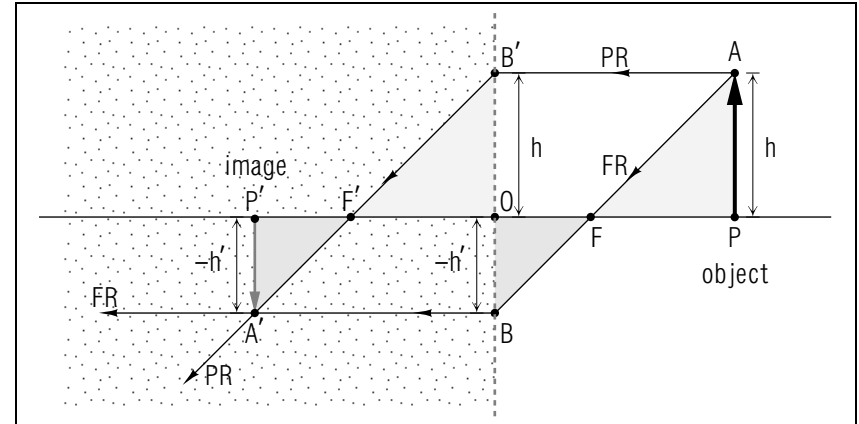
$$\theta + \alpha + (\pi - \gamma) = \pi \quad \text{and} \quad \theta' + \beta + (\pi - \gamma) = \pi,$$

so,

$$\theta = \gamma - \alpha \quad \text{and} \quad \theta' = \gamma - \beta \quad (10)$$

where

$$\sin \gamma = \frac{d}{x_C}, \quad \tan \alpha = \frac{d}{x_o - x_d}, \quad \text{and} \quad \tan \beta = \frac{d}{x_i - x_d}.$$



**Figure 9.** The similar triangles used to derive the image height given the object height.

These relations cannot, in general, be solved for the image position  $x_i$  in terms of common functions. They are, however, soluble if we assume that the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  are small (and that, therefore,  $\theta$  and  $\theta'$  are small). Then  $\tan \alpha \approx \alpha$ ,  $\tan \beta \approx \beta$ ,  $\sin \gamma \approx \gamma$ ,  $\sin \theta \approx \theta$ , and  $\sin \theta' \approx \theta'$ . Using this approximation (called the “paraxial approximation”), Snell’s law may be written as:

$$n\theta = n'\theta',$$

where

$$\theta = \gamma - \alpha = \frac{d}{x_C} - \frac{d}{x_o},$$

and

$$\theta' = \gamma - \beta = \frac{d}{x_C} - \frac{d}{x_i},$$

so

$$n \left( \frac{d}{x_C} - \frac{d}{x_o} \right) = n' \left( \frac{d}{x_C} - \frac{d}{x_i} \right).$$

This expression may be rearranged as:

$$\frac{n}{x_o} - \frac{n'}{x_i} = \frac{n - n'}{x_C} \quad (11)$$

which is Descartes’ formula for refraction at a single spherical surface.

## B. Derivation of the Image Height

The size of an image is determined by the point at which the focal ray strikes the refracting surface.<sup>5</sup> Triangles ( $AFP$ ) and ( $BFO$ ) in Fig. 9 are similar, so

$$\frac{OF}{BO} = \frac{PF}{AP}, \quad \text{or} \quad \frac{x_F}{-h'} = \frac{x_o - x_F}{h},$$

so:

$$h' = - \left( \frac{x_F}{x_o - x_F} \right) h. \quad (12)$$

This is Eq. (7).

---

<sup>5</sup>We will define  $h'$  as being positive if the image is erect and negative if the image is inverted.

## PROBLEM SUPPLEMENT

Note: Problems 4 and 5 also occur in this module's *Model Exam*.

- An object 5.0 cm tall is placed 2.0 m to the right of a convex surface whose radius of curvature is 0.50 m. The medium to the left of the surface has an index of refraction of 1.50, and that to the right of the surface has an index of refraction of 1.00.
  - Find the positions of the focal points. Is the system a converging or diverging one?
  - Find the position of the image. Is it real or virtual?
  - Find the size of the image and the magnification. Is the image erect or inverted?
  - Draw a ray diagram approximately to scale (the  $x$ - and  $y$ -axes need not have same scale), using two of the principal rays. Show the positions of the foci and the image.
- Repeat the calculations and diagram for a case similar to Problem 1, but with the indices of refraction switched. That is, the index of refraction on the left side of the surface is 1.00, and the index of refraction on the right side is 1.50.
- A cylindrical glass rod has an index of refraction of 1.3333, and is 1.0 m long. The right end is convex (i.e., bulges toward the right) and has a radius of curvature of 0.050 m. The left end is also convex (i.e., bulges toward the left) and has a radius of curvature of 0.100 m. It is surrounded on all sides by air ( $n = 1.00$ ). An object 0.010 m high is placed 0.50 m to the right of the right end of the cylinder. *Help: [S-1]*
  - Find the position of the image formed by the right end surface of the cylinder relative to that surface. Is it real or virtual?
  - Find the size of that image. Is it erect or inverted?
  - Using the image formed by the right end surface as the "object" seen by the left end surface, find the final image position relative to the left end surface. Is it real or virtual?
  - Find the size of the final image. Is it erect or inverted relative to the original object?



- e. Find the focal points of each of the two surfaces in the system and use them to draw a ray diagram, approximately to scale, showing both the first and final images.

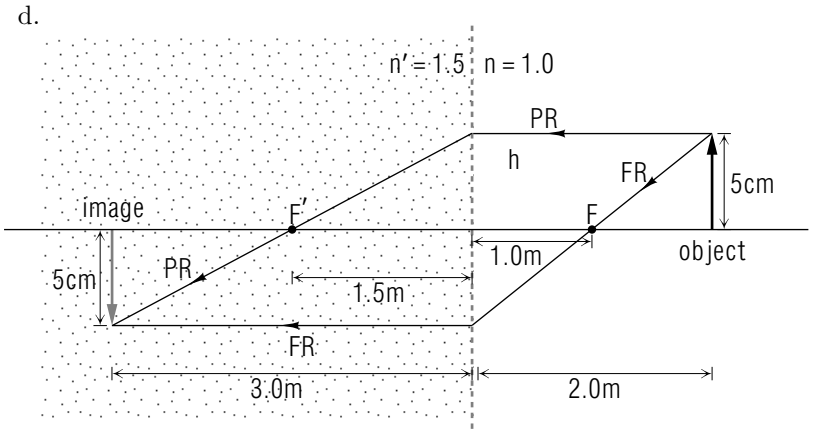
Notice that the system described in this problem is a very thick lens. The method used here is precisely the one used to deal with real lenses.<sup>6</sup>

4. An object 1.0 cm tall is 1.50 m to the right of a convex spherical surface whose radius of curvature is 0.50 m. The medium to the right of the surface is air (index of refraction = 1.00), and the medium to the left has an index of refraction equal to 1.67.
- Find the locations of the two focal points of the system.
  - Find the position of the image. Is it real or virtual?
  - Find the size of the image. Is it erect or inverted?
  - Construct the image by drawing a ray diagram, using two of the principal rays diverging from the head of the object. Indicate what happens to each of these rays after refraction. Label the focal points. Draw the distances to scale as well as you can.
5. Repeat the above calculations for the same system, only with the object being 0.50 m to the right of the surface.

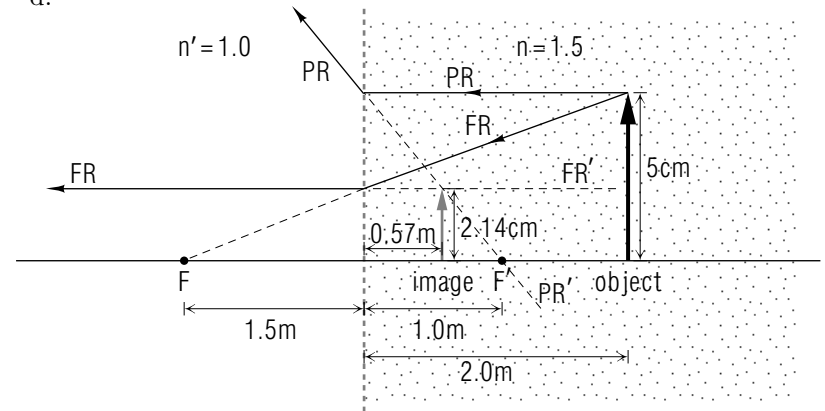
**Brief Answers:**

- All positions given here are relative to the refracting surface, with  $x_o = +2.0$  m,  $x_C = -0.50$  m,  $n = 1.0$ ,  $n' = 1.5$ .
  - $x_F = 1.0$  m is object focus position,  $x'_F = -1.5$  m is image focus position. System is converging.
  - $x_i = -3.0$  m (i.e., 3.0 m to left of surface). Image is real.
  - $h' = -5.0$  cm,  $M = -1.0$ . Image is inverted.

<sup>6</sup>See "Thin Spherical Lenses" (MISN-0-223).

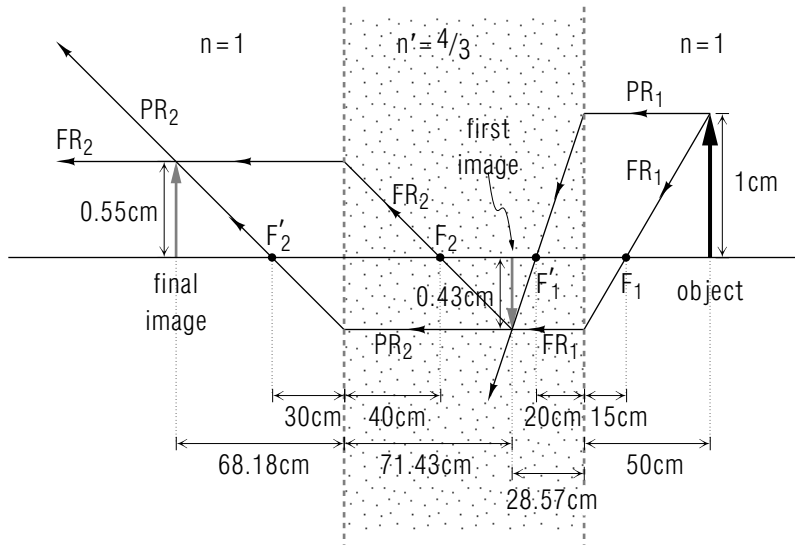


- Again, giving all positions relative to the refracting surface ( $x_o = +2.0$  m,  $x_C = -0.50$  m, but now  $n = 1.5$ ,  $n' = 1.0$ ).
  - $x_F = -1.5$  m is object focus position,  $x'_F = +1.0$  m is image focus position. System is diverging.
  - $x_i = +0.57$  m (0.57 m to right of surface). Image is virtual.
  - $h' = +2.1$  cm,  $M = +0.43$ . Image is erect.
  -



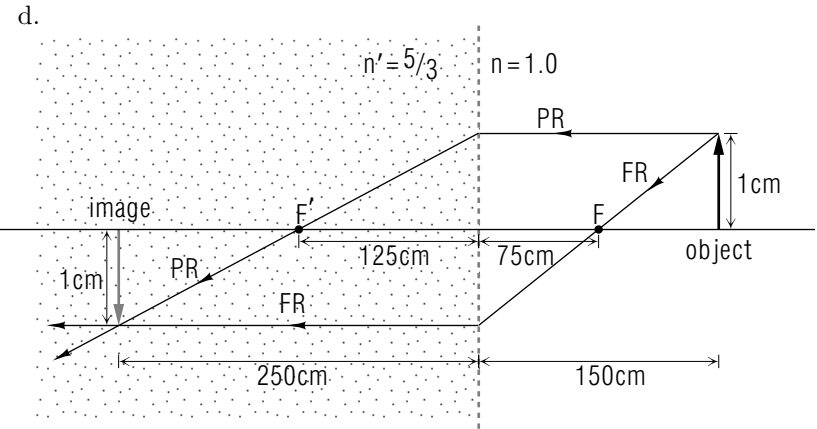
- Image is 29 cm to left of first surface. It is real.
  - $h = -0.43$  cm, image is inverted.
  - Final image is 68 cm to left of left surface. It is real.

- d.  $h = +0.55$  cm. Image is erect relative to original object.
  - e. Right surface: object focus is 15 cm to right of surface, image focus is 0.20 m to left of surface.
- Left surface: object focus is 0.40 m to right of surface, image focus is 0.30 m to left of surface.



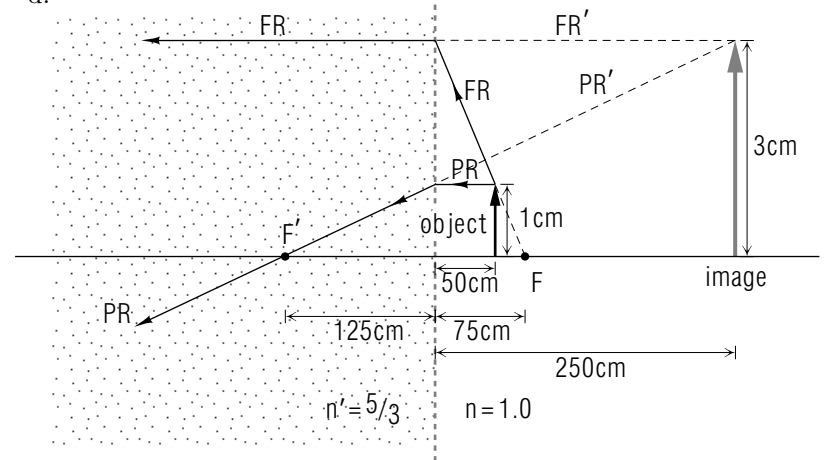
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- 4. Giving all positions relative to the refracting surface, with  $x$  increasing to the right ( $x_o = +1.5$  m,  $x_C = -0.50$  m,  $n = 1.0$ ,  $n' = 1.7$ ):
  - a.  $x_F = +75$  cm. Object focus 75 cm to right of surface;  $x'_F = -1.3$  m. Image focus 1.3 m to left of surface.
  - b.  $x_i = -2.5$  m. Image is 2.5 m to left of surface; image is real.
  - c.  $h' = -1.0$  cm. Image is inverted.



After refraction:  $PR$  passes through image focal point;  $FR$  travels parallel to axis;  $CR$  (unrefracted) passes through center of curvature of surface.

- 5. a.  $x_F = +75$  cm. Just the same as in previous case.  $x'_F = -1.3$  m.
- b.  $x_i = +2.5$  m. Image is 2.5 m to right of surface. Image is virtual.
- c.  $h' = +3.0$  cm. Image is erect.
- d.



After refraction:  $PR$  passes through image focal point;  $FR$  goes parallel to axis;  $CR$  (unrefracted) goes through center of curvature.

Virtual extensions of all three rays ( $PR'$ ,  $FR'$ , and  $CR'$ ) intersect at image point.

### SPECIAL ASSISTANCE SUPPLEMENT

**S-1**      (*from PS-3*)

If you have trouble visualizing this problem, note that the rod can be long and skinny (of narrow diameter). The diameter of the rod has no bearing on the position and size of the image. However, the diameter of the rod has a direct bearing on the brightness of the image since a larger end will simply have more light incident on it (we say it has more “light-gathering” power). Also see *Help: [S-2]*.

**S-2**      (*from PS-3 and Help: [S-1]*)

In the answer sketch, note that the vertical scale is very different from the horizontal scale. That is quite all right and it gives correct answers for all but the angles of the rays (which we do not ask you for).

**MODEL EXAM**

$$\frac{n}{x_o} - \frac{n'}{x_i} = \frac{n - n'}{x_C}$$

1. See Output Skills K1-K2 in this module's *ID Sheet*.
2. An object 1.0 cm tall is 1.50 m to the right of a convex spherical surface whose radius of curvature is 0.50 m. The medium to the right of the surface is air (index of refraction = 1.00), and the medium to the left has an index of refraction equal to 1.67.
  - a. Find the locations of the two focal points of the system.
  - b. Find the position of the image. Is it real or virtual?
  - c. Find the size of the image. Is it erect or inverted?
  - d. Construct the image by drawing a ray diagram, using two of the principal rays diverging from the head of the object. Indicate what happens to each of these rays after refraction. Label the focal points. Draw the distances to scale as well as you can.
3. Repeat the above calculations for the same system, only with the object being 0.50 m to the right of the surface.

**Brief Answers:**

1. See this module's *text*.
2. See Problem 4 in this module's *Problem Supplement*.
3. See Problem 5 in this module's *Problem Supplement*.