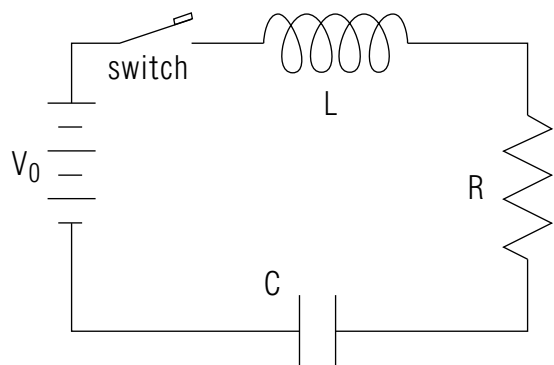


## THREE-ELEMENT DC-DRIVEN SERIES LRC CIRCUIT



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by  
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<b>1. Introduction and Description</b> .....	1
<b>2. Study Comments</b> .....	1
<b>Acknowledgments</b> .....	4

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**Input Skills:**

1. Skills from “Two-Element D.C.-Driven LRC Circuits” (MISN-0-151).

**Output Skills (Knowledge):**

- K1. Starting from the charge-current and voltage-current relations for the three types of passive circuit element:
  - a. Derive the relation between the time rate of change of charge and the circuit parameters.
  - b. Given a solution for the relation, evaluate as many constants as possible without using any information about the circuit’s initial state.
  - c. Explain why two solution forms are necessary for the relation.

**Post-Options:**

1. “The Driven LRC Circuit: Resonances” (MISN-0-154).

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### 1. Introduction and Description

In this module we analyze the special case of the D.C.-driven series *LRC* circuit. By “series *LRC*” we mean there is an inductor, a resistor, and a capacitor connected in series. By “D.C.-driven” we mean that a direct-current voltage is applied for some period of time and there is no other source of EMF (see Fig. 1.). We examine the mathematics used to find a complete solution for the time dependence of the circulating charge.

### 2. Study Comments

The general method of attack on the D.C.-driven three-element circuit is the same as that used in the two-element case.<sup>1</sup> The reason the 3-element circuit is treated separately is that there are two solutions. To see why, we work through the circuit shown in Fig. 1.

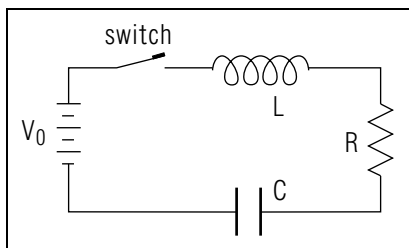
Following the same procedure used for the 2-element circuit, we easily arrive at the fundamental equation for this circuit:

$$V_\epsilon = L \frac{d^2 q(t)}{dt^2} + R \frac{dq(t)}{dt} + \frac{1}{C} q(t) \quad (1)$$

A solution of Eq. (1) is:

$$q(t) = q_0 + q_1 e^{-at} \sin(bt + d), \quad (2)$$

<sup>1</sup>See MISN-0-151.



**Figure 1.** Switch open for  $t < 0$ , closed for  $t > 0$ .

where  $q_0$ ,  $q_1$ ,  $a$ ,  $b$  and  $d$  are constants.

To prove that this is indeed a solution, we insert it and its first and second derivatives into Eq. (1). Collecting terms, we get:

$$V_\epsilon = \Lambda q_1 e^{-at} \sin(bt + d) + (Rb - 2Lab) q_1 e^{-at} \cos(bt + d) + \frac{q_0}{C}, \quad (3)$$

where we have defined for convenience:

$$\Lambda \equiv La^2 - Lb^2 - Ra + \frac{1}{C}.$$

Since Eq. (3) holds for all times  $t$ , we can evaluate the constants at any time we wish. To make our job easy we choose  $t$  such that  $e^{-at} = 0$  ( $t \rightarrow \infty$ ). We get:

$$V_\epsilon = \frac{q_0}{C},$$

so

$$q_0 = CV_\epsilon. \quad (4)$$

Putting this back into Eq. (3) and rearranging gives:

$$V_\epsilon - \Lambda q_1 e^{-at} \sin(bt + d) = V_\epsilon + (Rb - 2Lab) q_1 e^{-at} \cos(bt + d). \quad (5)$$

Cancelling terms,

$$-\Lambda \sin(bt + d) = (Rb - 2Lab) \cos(bt + d). \quad (6)$$

We now choose to evaluate Eq. (6) at  $t = -d/b$  so  $\sin(bt + d) = 0$  and  $\cos(bt + d) = 1$ , resulting in:

$$Rb - 2Lab = 0,$$

so:

$$a = \frac{R}{2L}. \quad (7)$$

After substituting this into (6), we pick  $t$  so that  $\sin(bt + d) = 1$  and  $\cos(bt + d) = 0$ , giving:

$$Lb^2 - \frac{R^2}{4L} + \frac{R^2}{2L} - \frac{1}{C} = 0.$$

Solving for  $b$  gives:

$$b = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \quad (8)$$

Here is what we have found: Eq. (2) is a solution to Eq. (1) providing  $q_0$ ,  $a$ , and  $b$  are restricted to certain combinations of the circuit parameters. However, Eq. (2) is a solution to Eq. (1) no matter what the values of  $q_1$  and  $d$ , so these latter are the two adjustable constants required in a solution of a second order differential equation. They must be set from the initial conditions for a particular problem you are wanting to solve.

The solution we have developed has a problem if  $R^2/4L^2 > 1/LC$ , for then the argument of the square root in Eq. (8) is negative! Then we must add a restriction to the solution we have found:

$$q(t) = CV_\epsilon + q_1 e^{-(R/2L)t} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} t + d\right); \quad R \leq 2\sqrt{L/C}. \quad (9)$$

Now what about cases that violate the restriction? For such cases we replace the circular functions, sine and cosine, with their hyperbolic counterparts:

$$\sinh(t) \equiv \frac{e^t - e^{-t}}{2} \quad \cosh(t) \equiv \frac{e^t + e^{-t}}{2}.$$

Note that the derivative of (cosh) is (+sinh), as opposed to the derivative of (cos) being (-sin) for the circular functions. We write this solution as:

$$q(t) = q_0 + q_1 e^{-at} \sinh(b't + d'),$$

where  $q_0$ ,  $q_1$ ,  $a$ ,  $b'$  and  $d'$  are constants. Substitution into Eq. (1) produces:

$$V_\epsilon = \Lambda' q_1 e^{-at} \sinh(b't + d') + (Rb' - 2Lab') q_1 e^{-at} \cosh(b't + d') + \frac{q_0}{C}. \quad (10)$$

where:

$$\Lambda' \equiv La^2 + Lb'^2 - Ra + \frac{1}{C}.$$

Substituting into Eq. (10) and cancelling yields:

$$\left(-Lb'^2 - \frac{R^2}{4L} + \frac{R^2}{2L} - \frac{1}{C}\right) \sinh(b't + d') = 0.$$

Since this holds for any  $t$ , and  $\sinh(b't + d')$  is not always zero, the coefficient of  $\sinh(b't + d')$  must be zero:

$$-Lb'^2 - \frac{R^2}{4L} + \frac{R^2}{2L} - \frac{1}{C} = 0.$$

Thus:

$$b' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}},$$

which makes our second solution:

$$q(t) = CV_\epsilon + q_1 e^{-(R/2L)t} \sinh\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} t + d'\right); \quad R \geq \sqrt{L/C}.$$

The unevaluated constants in our two solutions,  $q_1$  and either  $d$  or  $d'$ , cannot be determined without specific information about the circuit's initial state.

## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

**MODEL EXAM**

$$q(t) = q_0 + q_1 e^{-at} \sin(bt + d)$$

$$q(t) = q_0 + q_1 e^{-at} \sinh(b't + d')$$

1. See Output Skill K1 in this module's *ID Sheet*.

**Brief Answers:**

1. See this module's *text*.

