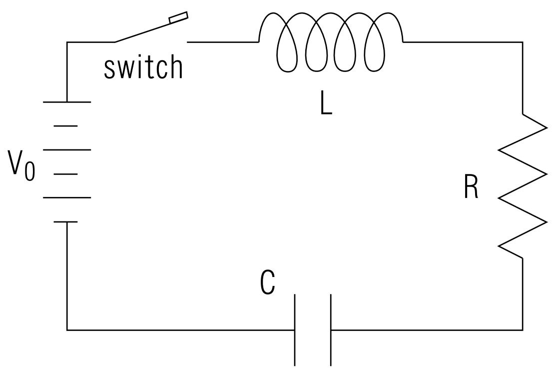
MISN-0-152

THREE-ELEMENT DC-DRIVEN SERIES LRC CIRCUIT



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

THREE-ELEMENT DC-DRIVEN SERIES LRC CIRCUIT

by

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Title: **Three-Element DC-Driven Series LRC Circuit**

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Length: 1 hr; 12 pages

Input Skills:

1. Skills from "Two-Element D.C.-Driven LRC Circuits" (MISN-0-151).

Output Skills (Knowledge):

K1. Starting from the charge-current and voltage-current relations for the three types of passive circuit element:
   a. Derive the relation between the time rate of change of charge and the circuit parameters.
   b. Given a solution for the relation, evaluate as many constants as possible without using any information about the circuit's initial state.
   c. Explain why two solution forms are necessary for the relation.

Post-Options:

1. "The Driven LRC Circuit: Resonances" (MISN-0-154).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

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1. Introduction and Description

In this module we analyze the special case of the D.C.-driven series LRC circuit. By "series LRC" we mean there is an inductor, a resistor, and a capacitor connected in series. By "D.C.-driven" we mean that a direct-current voltage is applied for some period of time and there is no other source of EMF (see Fig. 1.). We examine the mathematics used to find a complete solution for the time dependence of the circulating charge.

2. Study Comments

The general method of attack on the D.C.-driven three-element circuit is the same as that used in the two-element case.[1] The reason the 3-element circuit is treated separately is that there are two solutions. To see why, we work through the circuit shown in Fig. 1.

Following the same procedure used for the 2-element circuit, we easily arrive at the fundamental equation for this circuit:

$$V_\epsilon = L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}q(t) \tag{1}$$

A solution of Eq. (1) is:

$$q(t) = q_0 + q_1 e^{-at}\sin(bt+d), \tag{2}$$

[1]See MISN-0-151.

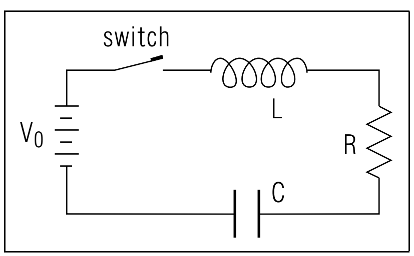

Figure 1. Switch open for $t < 0$, closed for $t > 0$.

where q_0, q_1, a, b and d are constants.

To prove that this is indeed a solution, we insert it and its first and second derivatives into Eq. (1). Collecting terms, we get:

$$V_\epsilon = \Lambda q_1 e^{-at}\sin(bt+d) + (Rb - 2Lab)q_1 e^{-at}\cos(bt+d) + \frac{q_0}{C}, \tag{3}$$

where we have defined for convenience:

$$\Lambda \equiv La^2 - Lb^2 - Ra + \frac{1}{C}.$$

Since Eq. (3) holds for all times t, we can evaluate the constants at any time we wish. To make our job easy we choose t such that $e^{-at} = 0$ $(t \to \infty)$. We get:

$$V_\epsilon = \frac{q_0}{C},$$

so

$$q_0 = CV_\epsilon. \tag{4}$$

Putting this back into Eq. (3) and rearranging gives:

$$V_\epsilon - \Lambda q_1 e^{-at}\sin(bt+d) = V_\epsilon + (Rb - 2Lab)q_1 e^{-at}\cos(bt+d). \tag{5}$$

Cancelling terms,

$$-\Lambda\sin(bt+d) = (Rb - 2Lab)\cos(bt+d). \tag{6}$$

We now choose to evaluate Eq. (6) at $t = -d/b$ so $\sin(bt+d) = 0$ and $cos(bt+d) = 1$, resulting in:

$$Rb - 2Lab = 0,$$

so:

$$a = \frac{R}{2L}. \tag{7}$$

After substituting this into (6), we pick t so that $\sin(bt+d) = 1$ and $\cos(bt+d) = 0$, giving:

$$Lb^2 - \frac{R^2}{4L} + \frac{R^2}{2L} - \frac{1}{C} = 0.$$

Solving for b gives:

$$b = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \tag{8}$$

Here is what we have found: Eq. (2) is a solution to Eq. (1) providing q_0, a, and b are restricted to certain combinations of the circuit parameters. However, Eq. (2) is a solution to Eq. (1) no matter what the values of q_1 and d, so these latter are the two adjustable constants required in a solution of a second order differential equation. They must be set from the initial conditions for a particular problem you are wanting to solve.

The solution we have developed has a problem if $R^2/4L^2 > 1/LC$, for then the argument of the square root in Eq. (8) is negative! Then we must add a restriction to the solution we have found:

$$q(t) = CV_\epsilon + q_1 e^{-(R/2L)t} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t + d\right); \;\; R \le 2\sqrt{L/C}. \quad (9)$$

Now what about cases that violate the restriction? For such cases we replace the circular functions, sine and cosine, with their hyperbolic counterparts:

$$\sinh(t) \equiv \frac{e^t - e^{-t}}{2} \qquad \cosh(t) \equiv \frac{e^t + e^{-t}}{2}.$$

Note that the derivative of (cosh) is $(+\sinh)$, as opposed to the derivative of (cos) being $(-\sin)$ for the circular functions. We write this solution as:

$$q(t) = q_0 + q_1 e^{-at} \sinh(b't + d'),$$

where q_0, q_1, a, b' and d' are constants. Substitution into Eq. (1) produces:

$$V_\epsilon = \Lambda' q_1 e^{-at} \sinh(b't + d') + (Rb' - 2Lab') q_1 e^{-at} \cosh(b't + d') + \frac{q_0}{C}. \quad (10)$$

where:

$$\Lambda' \equiv La^2 + Lb'^2 - Ra + \frac{1}{C}.$$

Substituting into Eq. (10) and cancelling yields:

$$\left(-Lb'^2 - \frac{R^2}{4L} + \frac{R^2}{2L} - \frac{1}{C}\right) \sinh(b't + d') = 0.$$

Since this holds for any t, and $\sinh(b't + d')$ is not always zero, the coefficient of $\sinh(b't + d')$ must be zero:

$$-Lb'^2 - \frac{R^2}{4L} + \frac{R^2}{2L} - \frac{1}{C} = 0.$$

Thus:

$$b' = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\,,$$

which makes our second solution:

$$q(t) = CV_\epsilon + q_1 e^{-(R/2L)t} \sinh\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\, t + d'\right) ; \;\; R \geq \sqrt{L/C}.$$

The unevaluated constants in our two solutions, q_1 and either d or d', cannot be determined without specific information about the circuit's initial state.

Acknowledgments

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MODEL EXAM

$$q(t) = q_0 + q_1 e^{-at} \sin(bt + d)$$
$$q(t) = q_0 + q_1 e^{-at} \sinh(b't + d')$$

1. See Output Skill K1 in this module's *ID Sheet*.

Brief Answers:

1. See this module's *text*.